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Abstract

Full Text

MATHEMATICS

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ON SMALL RANDOM PERTURBATIONS OF A DYNAMICAL SYSTEM WITH A STABLE EQUILIBRIUM POSITION

(Presented by Academician A. N. Kolmogorov, 30 XII 1968)

Consider a dynamical system in R^n :

$$\dot{x} = b(x), \quad (1)$$

for which the origin is a point of stable equilibrium. Let D be a bounded domain containing the point zero, and let Γ be the boundary of the domain D . Suppose that the trajectories of system (1) issuing from points $x \in D$ enter the origin without leaving the domain D . Consider the Markov process x_t^ε , which is obtained from (1) if the right-hand side is perturbed by white noise ξ_t with a small coefficient ε ⁽²⁾:

$$dx_t^\varepsilon = \varepsilon d\xi_t + b(x_t^\varepsilon) dt. \quad (2)$$

For every $\varepsilon > 0$, the trajectories of the process x_t^ε leave the domain D with probability one and induce on Γ a certain distribution $\pi^\varepsilon(x, \gamma) = P_x\{x_{\tau_\varepsilon}^\varepsilon \in \gamma\}$, where $\gamma \subset \Gamma$, $x = x_0^\varepsilon$, $\tau_\varepsilon = \inf\{t : x_t^\varepsilon \in D\}$.

The purpose of the present note is to study the ways in which the trajectories of the process x_t^ε (and of some more general processes) exit to the boundary Γ of the domain D as $\varepsilon \rightarrow 0$. It turns out that under certain natural assumptions the distribution $\pi^\varepsilon(x, \cdot)$ as $\varepsilon \rightarrow 0$ concentrates in a neighborhood of a single point on Γ . Moreover, the last segment before reaching Γ , but after leaving a neighborhood of the equilibrium position, is traversed by the trajectories near a fixed curve φ_s , an extremal of a certain functional.

This problem, in the case of perturbations of a somewhat more general form, can be formulated as follows. Consider in the domain D the Dirichlet problem

$$L^\varepsilon u^\varepsilon(x) = \frac{\varepsilon^2}{2} \sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x) \frac{\partial u^\varepsilon}{\partial x_i}, \quad u^\varepsilon(x)|_\Gamma = \psi(x), \quad (3)$$

where the form

$$\sum_{i,j=1}^n a_{ij}(x) \lambda_i \lambda_j$$

is positive definite, $a_{ij}(x), b_i(x) \in C^1(R^n)$. Does $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x)$ exist, and what is it equal to? Denote by $\{a^{ij}(x)\}$ the matrix inverse to $a(x) = \{a_{ij}(x)\}$, and put

$$I(\varphi) = I_{T_1, T_2}(\varphi) = \int_{T_1}^{T_2} \sum_{i,j=1}^n a^{ij}(\varphi_s) (\dot{\varphi}_i - b_i(\varphi_s)) (\dot{\varphi}_j - b_j(\varphi_s)) ds.$$

This functional is considered on the class H_{T_1, T_2} of continuous functions φ_s , $s \in [T_1, T_2]$, with values in R^n , having piecewise-continuous derivative $\dot{\varphi}_s = (\dot{\varphi}_1, \dots, \dot{\varphi}_n)$.

The quasipotential of the vector field $b(x) = \{b_i(x)\}$ with respect to the form

$$\sum_{i,j=1}^n a^{ij}(x) \lambda_i \lambda_j$$

will be called the function $V(x)$, defined by the equal—

$$V(x) = \frac{1}{4} \inf_{\substack{\varphi_s \in H_T, T_1, T_2, T_1 \leq T_2 \\ \varphi_{T_1} = 0, \varphi_{T_2} = x}} I(\varphi).$$

It is easy to verify that the quasipotential is continuous in x , $V(0) = 0$, and $V(x) > 0$ for $x \neq 0$. We note that the quasipotential can be defined by the lower bound over the set of functions $H_T = \bigcup_{T \geq 0} H_{0, T}$; however, in general, $\inf I(\varphi)$ may be attained when $T_1 = -\infty$.

Theorem 1. *Suppose that there exists a unique point $x_0 \in \Gamma$ such that $V(x_0) = \min_{x \in \Gamma} V(x)$. Then $\lim_{\varepsilon \rightarrow 0} u^\varepsilon(x)$ exists and is equal to $\psi(x_0)$.*

Suppose that the field $b(x)$ can be represented in the form

$$b(x) = -a(x) \nabla U(x) + l(x),$$

where $U(x) \in C^1(D \cup \Gamma)$, $U(0) = 0$, $\nabla U(x) \neq 0$ for $x \neq 0$, and the vector field

$l(x)$ is orthogonal to $\nabla U(x)$: $(l(x), \nabla U(x)) = 0$. Then the equality

$$\begin{aligned}
 I_{T_1, T_2}(\varphi) &= \int_{T_1}^{T_2} (a^{-1}(\varphi_s)(\dot{\varphi}_s - a(\varphi_s)\nabla U(\varphi_s) - l(\varphi_s)), \\
 &\quad \dot{\varphi}_s - a(\varphi_s)\nabla U(\varphi_s) - l(\varphi_s)) ds + 4 \int_{T_1}^{T_2} (\dot{\varphi}_s, \nabla U(\varphi_s)) ds \quad (4) \\
 &\geq 4[U(\varphi_{T_2}) - U(\varphi_{T_1})].
 \end{aligned}$$

On the other hand, taking into account that the curve $\hat{\varphi}_s$, defined by the equation

$$\dot{\hat{\varphi}} = a(\hat{\varphi})\nabla U(\hat{\varphi}) + l(\hat{\varphi}), \quad \hat{\varphi}(T) = x,$$

enters, as $s \rightarrow -\infty$, any neighborhood of the equilibrium position (since

$$\frac{dU(\hat{\varphi}_s)}{ds} = |\nabla U(\varphi_s)|^2 > 0$$

for $\hat{\varphi} \neq 0$), we conclude that $I_{-\infty, T}(\hat{\varphi}) = 4U(x)$. Thus, from the last equality and (4), the following theorem can be derived.

Theorem 2. *Suppose that the field $b(x)$ admits the decomposition*

$$b(x) = -a(x)\nabla U(x) + l(x),$$

where $U(x) \in C^1(D \cup \Gamma)$, $U(0) = 0$, $U(x) > 0$ for $x \neq 0$, and $(l(x), \nabla U(x)) = 0$. Then the quasipotential coincides with $U(x)$. In particular, if the field $b(x)$ is potential and $\{a_{ij}(x)\}$ is the identity matrix, then the quasipotential coincides with the potential. If $U(x) \in C^2(D \cup \Gamma)$, then the unique extremal of the functional $I(\varphi)$ on the set $\{\varphi : \varphi_{T_2} = x\}$ is given by the equation

$$\dot{\varphi}_s = a(\varphi_s)\nabla U(\varphi_s) + l(\varphi_s), \quad s \in (-\infty, T_2], \quad \varphi_{T_2} = x.$$

Remark 1. From Theorem 2 it follows that the quasipotential can be defined as a nonnegative solution of the problem

$$\begin{aligned}
 (a(x)\nabla U(x), \nabla U(x)) + (b(x), \nabla U(x)) &= 0, \\
 U(0) = 0, \quad \nabla U(x) &\neq 0 \quad \text{for } x \neq 0.
 \end{aligned} \quad (5)$$

It is more natural to regard the function $V(x)$ as a generalized solution of this problem. If $V(x)$ is sufficiently smooth, then it satisfies equation (5).

In the case of the identity matrix $\{a_{ij}(x)\}$ and a potential field $b(x)$, R. Z. Khasminskii found the limit $u^\varepsilon(x)$ as $\varepsilon \rightarrow 0$, using an explicit expression for the density of the invariant measure of the process X^ε (unpublished). He also put forward a hypothesis concerning the nonpotential case: if a solution of problem (5) exists, then $\lim U^\varepsilon(x) = \psi(x_0)$, where $x_0 \in \Gamma$ is the point of minimum of

$U(x)$ on Γ , assumed to be unique. As follows from Remark 1, this hypothesis is valid.

Remark 2. The assertion of Theorem 1 remains valid for the solution of a problem of the form

$$\tilde{L}^\varepsilon u^\varepsilon(x) = \frac{\varepsilon^2}{2} \sum_{i,j=1}^n a_{ij}^\varepsilon(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_i} = 0, \quad u^\varepsilon(x)|_\Gamma = \psi(x),$$

provided only that $a_{ij}^\varepsilon(x) \rightarrow a_{ij}(x)$, $b_i^\varepsilon(x) \rightarrow b_i(x)$ as $\varepsilon \rightarrow 0$ uniformly in $x \in D \cup \Gamma$.

Let $X^\varepsilon = \{x_t^\varepsilon, P_x\}$ denote the Markov process governed by the operator L^ε (see (3)). The assertion of Theorem 1 is a consequence of the nature of the behavior of the trajectories of the process X^ε as $\varepsilon \rightarrow 0$. It turns out that, for small ε , the trajectories, with probability close to 1, enter a neighborhood of the equilibrium position. Further, the trajectories visit neighborhoods of every point x for which $V(x) < V(x_0) = \inf_{x \in \Gamma} V(x)$, returning many times to a neighborhood of zero. The first exit to the boundary of the domain D occurs in a neighborhood of the point $x_0 \in \Gamma$. An even stronger assertion is valid: the last portion of the trajectory before reaching the boundary, but after the last exit from a neighborhood of zero, lies near an extremal of the functional $I_{T_1 T_2}(\varphi)$ (more precisely, in a neighborhood of the set of functions φ for which $I(\varphi)$ differs little from $V(x_0)$).

Such behavior of the trajectories is a consequence of Lemmas 1-3. Introduce in the space C_T of continuous functions on the interval $[0, T]$ with values in R^n the distance

$$\rho_T(x, y) = \max_{0 \leq s \leq T} |x_s - y_s|.$$

Lemma 1. For any $K > 0$, $h > 0$, and any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \leq \varepsilon_0$ and any $x \in D$,

$$P_x \{ \rho_T(x^\varepsilon, \varphi) < \delta \} > \exp\{-I(\varphi)/2\varepsilon^2 - h/2\varepsilon^2\},$$

where φ_s is any continuously differentiable function from C_T , $\varphi_0 = x$, such that

$$\int_0^T (\dot{\varphi}_s) ds < K.$$

Lemma 2. Let I_1 be some positive constant. For any $h > 0$ and any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \leq \varepsilon_0$ and any $x \in D$,

$$P_x\{\rho(x^\varepsilon, \Phi_1) > \delta\} < \exp\{-I_1/2\varepsilon^2 + h/2\varepsilon^2\},$$

where Φ_1 is the set of functions $\varphi \in C_T$, $\varphi_0 = x$, possessing a piecewise continuous derivative, for which $I(\varphi) \leq I_1$.

Lemma 3. Suppose that all trajectories of system (1) issuing from $x \in D \cup \Gamma$ enter any neighborhood of the equilibrium position. Then, for any $\lambda > 0$, there exist constants $\varepsilon_0, c, T_0 > 0$ such that, for all $x \in D$, $\varepsilon_0 > \varepsilon$,

$$P_x\left\{\min_{0 \leq s \leq T} |x_s^\varepsilon| > \lambda\right\} \leq \exp\left\{-\frac{c(T - T_0)}{2\varepsilon^2}\right\}. \quad (6)$$

Let us explain the proof of Lemmas 1 and 2. To avoid cumbersome formulas, we restrict ourselves to the case of the identity matrix $\{a_{ij}(x)\}$.

For the proof of Lemma 1, one uses the absolute continuity of measures in the space of trajectories corresponding to diffusion processes differing only in the drift (1). The formula holds

$$\begin{aligned} & P_x\{\rho_T(x^\varepsilon, \varphi) < \delta\} = \\ & = \exp\left\{-\frac{I(\varphi)}{2\varepsilon^2}\right\} M_x \chi_{\{\rho_T(\xi_s^\varepsilon, 0)\}} \exp\left\{\frac{1}{\varepsilon} \int_0^T [b(\varphi_s + \varepsilon\xi_s) + \dot{\varphi}_s] d\xi_s + \right. \\ & \left. + \frac{1}{2\varepsilon^2} \int_0^T (|b(\varphi_s + \varepsilon\xi_s) - \dot{\varphi}_s|^2 - |b(\varphi_s) - \dot{\varphi}_s|^2) ds\right\}. \quad (7) \end{aligned}$$

It can be verified that the mathematical expectation in the right-hand side of inequality (7) is not less than $\exp\{-h/2\varepsilon^2\}$, provided only that ε is sufficiently small. Lemma 1 follows from this.

For the proof of Lemma 2, take small $r > 0$ and $\Delta_0 > 0$, and introduce random times τ_i , defined recursively: $\tau_0 = 0$, $\tau_{i+1} = \tau_i + \inf\{t : |\xi_{\tau_i+t} - \xi_{\tau_i}| = r/\varepsilon\}$, if this lower bound is less than Δ_0 , and $\tau_{i+1} = \tau_i + \Delta_0$ otherwise. Consider the polygonal line y_t , close to x_t^ε , with vertices at the points $(\tau_i, x_{\tau_i}^\varepsilon)$. The principal part of the functional $I(y_t)$ does not exceed

$$\sum_{\tau_i < T} \frac{r^2}{\tau_{i+1} - \tau_i}.$$

The probability in the left-hand side of (6) is estimated by means of the inequality

$$P_x\{I(y_t) > I_1\} \leq M_x \exp\left\{\frac{1-\lambda}{2\varepsilon^2}I(y_t)\right\} / \exp\left\{\frac{1-\lambda}{2\varepsilon^2}I_1\right\},$$

where $\lambda > 0$, and, for estimating the numerator, the fact is used that $\tau_{i+1} - \tau_i$ are independent random variables with a known distribution.

Let now the field $b(x)$ be arbitrary. Consider the boundary-value problem

$$\begin{aligned} \partial u^\varepsilon(t, x) / \partial t &= L^\varepsilon u^\varepsilon(t, x) + c(x)u^\varepsilon(t, x) \quad \text{for } t > 0, \quad x \in D, \\ u^\varepsilon(0, x) &= 0, \quad u^\varepsilon(t, x)|_{x \in \Gamma} = \psi(x), \end{aligned} \quad (8)$$

where $c(x)$ and $\psi(x)$ are continuous functions. From Lemmas 1 and 2 it follows

Theorem 3. Denote by $v^\varepsilon(t, x)$ the solution of problem (8) for $c(x) \equiv 0$, $\psi(x) \equiv 1$. Put

$$I_0(t, x) = \inf I_{0,t}(\varphi),$$

where the lower bound is taken over all functions φ such that $\varphi_0 = x$ and φ_s reaches the boundary before time t . Then:

a)

$$\lim_{\varepsilon \rightarrow 0} 2\varepsilon^2 \ln v^\varepsilon(t, x) = -I_0(t, x);$$

b) if the minimum $I_0(t, x)$ of the functional $I_{0,t}(\varphi)$ is positive and is attained for $\varphi = \hat{\varphi}_s$, $s \in [0, t]$, and

$$I_{0,t}(\varphi) > I_{0,t}(\hat{\varphi}) + \lambda(h) \quad \text{when } \rho_t(\hat{\varphi}, \varphi) > h,$$

where $\lim_{h \rightarrow 0} \lambda(h) = 0$, $\lambda(h) > 0$, then

$$\lim_{\varepsilon \rightarrow 0} \frac{u^\varepsilon(t, x)}{v^\varepsilon(t, x)} = \psi(\hat{\varphi}_{\hat{\tau}}) \exp\left\{\int_0^{\hat{\tau}} c(\hat{\varphi}_s) ds\right\},$$

where $\hat{\tau}$ is the first time at which $\hat{\varphi}_s$ reaches the boundary.

An analogous theorem can be formulated for the solution of the Cauchy problem with an initial function that vanishes outside a certain domain.

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Note: Figure translations are in progress. See original paper for figures.

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