



Soviet-era science, translated into English

INDEPENDENCE OF SUBSEMIGROUPS OF A SEMIGROUP

MATHEMATICS

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.13689>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.45

MATHEMATICS

E. S. LYAPIN

INDEPENDENCE OF SUBSEMIGROUPS OF A SEMIGROUP

(Presented by Academician Yu. V. Linnik on 23 IX 1968)

1°. The object of study of the present paper is the relation of independence of subsemigroups of a semigroup. The corresponding concept is quite natural and clearly deserves study. It may be regarded as a natural analogue of the idea arising in group theory in defining the very important concept there of a free decomposition of a group. However, the following circumstance is found. Whereas in group theory there are several approaches to the concept of a free decomposition which are equivalent to one another, in semigroup theory the corresponding concepts are no longer equivalent. One of them leads to the concepts considered below of independence of subsemigroups and decomposition of semigroups into independent products. Further investigation of these concepts leads to a theory substantially different from the theory of free decompositions in its usual sense. This is directly connected with the fact that the transfer beyond the limits of group theory of the theory of free decompositions developed there, successfully carried out for a number of classes of algebras (see (1)), has not in fact been accomplished for the class of semigroups. We note that, in connection with the study of the concept of independence of subsemigroups that is basic for this article, there appears the auxiliary concept of a unit-ideal subsemigroup, which may also be of independent interest, being a new generalization of the important concept of an ideal.

2°. **Definition.** A nonempty subset \mathfrak{K} of a semigroup \mathfrak{A} will be called a **unit-ideal subsemigroup** of the semigroup \mathfrak{A} if, for all $z \in \mathfrak{K}$ and $a \in \mathfrak{A}$, the product az is equal to a or belongs to \mathfrak{K} , and the product za is equal to a or belongs to \mathfrak{K} .

3°. Let us indicate several of the simplest properties of unit-ideal subsemigroups of a semigroup \mathfrak{A} .

- (α) A unit-ideal subsemigroup is indeed a subsemigroup.
- (β) Every ideal \mathfrak{A} is its unit-ideal subsemigroup (in particular, \mathfrak{A} itself).
- (γ) A subset consisting of a single unit element of \mathfrak{A} is a unit-ideal subsemigroup.

(δ) If \mathfrak{K} is such a subsemigroup of \mathfrak{A} that $\mathfrak{A} \setminus \mathfrak{K}$ consists of one element, then \mathfrak{K} is unit-ideal.

(ε) The union of any family of unit-ideal subsemigroups is a unit-ideal subsemigroup.

(η) The intersection of any family of unit-ideal subsemigroups, if it is nonempty, is a unit-ideal subsemigroup.

(ζ) A group has no unit-ideal subsemigroups other than itself and the identity subsemigroup.

In connection with properties (ε) and (η), we note that the general theory of systems of subsets with such properties, developed in (3), No. 2, Ch. IV, is applicable to unit-ideal subsemigroups.

4°. Let \mathfrak{K} be a unitary ideal subsemigroup of the semigroup \mathfrak{A} . Denote by $\mathfrak{N}_{\mathfrak{A}}^l$ the set of all $x \in \mathfrak{K}$ such that $ax \in \mathfrak{K}$ for all $a \in \mathfrak{A}$, and by $\mathfrak{N}_{\mathfrak{A}}^r$ the set of all $x \in \mathfrak{K}$ such that $xa \in \mathfrak{K}$.

We shall say that \mathfrak{K} satisfies condition 4° if, for all $u \in \mathfrak{K} \setminus \mathfrak{N}_{\mathfrak{A}}^l$ and $b \in \mathfrak{A} \setminus \mathfrak{K}$, one has $bu = b$, and for all $v \in \mathfrak{K} \setminus \mathfrak{N}_{\mathfrak{A}}^r$ and $b \in \mathfrak{A} \setminus \mathfrak{K}$, one has $vb = b$.

5°. **Theorem 1.** *If a unitary ideal subsemigroup \mathfrak{K} of the semigroup \mathfrak{A} satisfies condition 4°, then $\mathfrak{N}_{\mathfrak{A}}^l$ is a left ideal of \mathfrak{A} or is empty, and $\mathfrak{N}_{\mathfrak{A}}^r$ is a right ideal of \mathfrak{A} or is empty.*

6°. In connection with 5° we note that, in the case under consideration, each element $b \in \mathfrak{A} \setminus \mathfrak{K}$ induces in $\mathfrak{N}_{\mathfrak{A}}^l$ a left-shift transformation, for which we fix the notation $\varphi_b^{\mathfrak{A}}$ ($\varphi_b^{\mathfrak{A}}x = bx$; $x \in \mathfrak{N}_{\mathfrak{A}}^l$).

In what follows we shall regard $\varphi_b^{\mathfrak{A}}$ as an element of the semigroup of all transformations of the set $\mathfrak{N}_{\mathfrak{A}}^l$. Similarly, the element b induces in $\mathfrak{N}_{\mathfrak{A}}^r$ a right-shift transformation, which we shall denote by $\psi_b^{\mathfrak{A}}$ ($\psi_b^{\mathfrak{A}}x = xb$; $x \in \mathfrak{N}_{\mathfrak{A}}^r$).

7°. Let \mathfrak{M} be some nonempty subset of the semigroup \mathfrak{A} . A pair of words over \mathfrak{M} , joined by an equality sign,

$$x_1x_2 \dots x_p = y_1y_2 \dots y_q$$

is called a relation over \mathfrak{M} if the values of these words in \mathfrak{A} are equal.

Let Γ be some set of subsemigroups of the semigroup \mathfrak{A} . We shall call a relation over \mathfrak{M} reduced with respect to Γ if no subsemigroup $\mathfrak{A}_{\xi} \in \Gamma$ contains simultaneously two adjacent x_k and x_{k+1} , or two adjacent y_l and y_{l+1} .

8°. **Definition.** A set of subsemigroups $\Gamma = \{\mathfrak{A}_{\xi}\}_I$ of the semigroup \mathfrak{A} will be called **independent** if no \mathfrak{A}_{ξ_0} ($\xi_0 \in I$) is contained in the union of the remaining \mathfrak{A}_{ξ} , and if on the set $\bigcup_I \mathfrak{A}_{\xi}$ there are no nonidentical relations reduced with respect to Γ .

If the set Γ is independent and $\bigcup_I \mathfrak{A}_\xi$ is a generating set in the semigroup \mathfrak{A} , then we shall say that \mathfrak{A} decomposes into an independent product of subsemigroups from Γ .

It is obvious that, for an independent set Γ , the subsemigroup generated by it, $\mathfrak{A}' = [\mathfrak{A}_\xi]_I$, decomposes into an independent product of subsemigroups from Γ .

9°. A group is a free product (in the sense of group theory) of a set of its subgroups Γ if and only if it is their independent product in the sense of 8°. Incidentally, let us mention that the condition that none of the components of a free decomposition should be contained in the union of the others (or some essentially analogous condition) must also be imposed in the definition of a free product in group theory, although this is often not done explicitly.

10°. Free products of groups with a common nonunit subgroup are never independent. At the same time, for any semigroup \mathfrak{H} one can find a semigroup \mathfrak{A} which is an independent product of certain subsemigroups \mathfrak{A}_1 and \mathfrak{A}_2 , with $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \mathfrak{H}$.

This shows especially convincingly that the construction of the independent product within semigroup theory is essentially different from the free product, as it is usually formulated by analogy with the way this is done in group theory.

11°. Let a semigroup amalgam be given, i.e. an arbitrary system of semigroups $\Gamma = \{\mathfrak{A}_\xi\}_I$, which may have various pairwise intersections. It is assumed here that the operations in the semigroups agree on their intersections. In full generality the question of when such a semigroup amalgam can be embedded in a semigroup appears at present to be too difficult. It is known that this is possible not always, even for a system consisting of two semigroups, and even when they are groups (see (2), (3), Ch. X, § 5; correcting the misprint there presents no difficulty).

It is true, as is known from group theory (see, for example, (4)), that in the case where the intersection of these two groups is a group, the amalgam is always embeddable in a group.

A number of investigations relating to the problem of semigroup amalgams were carried out by Howie (5-7).

12°. The question of when there exists an oversemigroup \mathfrak{A} of the semigroups \mathfrak{A}_ξ ($\xi \in I$), decomposing into their independent product, may be regarded as a special case of embedding a semigroup amalgam in a semigroup.

If such an oversemigroup \mathfrak{A} exists for a given system, then it is essentially unique.

13°. **Theorem 2.** *Let \mathfrak{K} be the intersection of two semigroups \mathfrak{A}_1 and \mathfrak{A}_2 , whose operations are compatible, i.e. coincide on the elements of \mathfrak{K} . In order that there exist an oversemigroup \mathfrak{A} of the semigroups \mathfrak{A}_1 and \mathfrak{A}_2 , which is their independent product, it is necessary and sufficient that \mathfrak{K} be empty or distinct from both \mathfrak{A}_1 and \mathfrak{A}_2 and satisfy the following three conditions:*

- 1) \mathfrak{K} is a two-sided ideal subsemigroup satisfying condition 4°, both for \mathfrak{A}_1 and for \mathfrak{A}_2 ;
- 2) $\mathfrak{R}_1^l = \mathfrak{R}_2^r$, $\mathfrak{R}_1^r = \mathfrak{R}_2^l$;
- 3) For any $b_1 \in \mathfrak{A}_1 \setminus \mathfrak{K}$ and $b_2 \in \mathfrak{A}_2 \setminus \mathfrak{K}$, if $\mathfrak{R}_1^l = \mathfrak{R}_2^r \neq \emptyset$, the following must hold:

$$\varphi_{b_1}^{(1)} \psi_{b_2}^{(2)} = \psi_{b_2}^{(2)} \varphi_{b_1}^{(1)},$$

and if $\mathfrak{R}_1^r = \mathfrak{R}_2^l \neq \emptyset$:

$$\psi_{b_1}^{(1)} \varphi_{b_2}^{(2)} = \varphi_{b_2}^{(2)} \psi_{b_1}^{(1)}.$$

(Here, for \mathfrak{R} , φ , ψ , instead of the index \mathfrak{A}_i , the index i is used.)

14°. In connection with condition (3) of 13°, we note that for a semigroup \mathfrak{H} such that $\mathfrak{H} \cdot \mathfrak{H} = \mathfrak{H}$, every left shift is always permutable with every right shift.

Leningrad State Pedagogical Institute
named after A. I. Herzen

Received
12 IX 1968

REFERENCES

1. T. M. Baranovich, *Sibirsk. matem. zhurn.*, **7**, No. 6, 1230 (1966).
2. N. Kimura, Doctoral Thesis, Tulane Univ., 1957.
3. E. S. Lyapin, *Semigroups*, Moscow, 1960.
4. A. G. Kurosh, *Group Theory*, 3rd ed., "Nauka," 1967.
5. J. M. Howie, *Proc. London Math. Soc.*, **12**, No. 47, 511 (1962).
6. J. M. Howie, *Proc. London Math. Soc.*, **13**, No. 52, 672 (1963).
7. J. M. Howie, *Proc. Glasgow Math. Assoc.*, **6**, 19 (1963).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.