THE GROUP OF PROJECTIVE TRANSFORMATIONS IN A COMPLETE ANALYTIC RIEMANNIAN SPACE

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Abstract

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1°. By $P(V)$ and $A(V)$ we denote (respectively) the group of all projective and the group of all affine transformations in the Riemannian space $* V$. Obviously, $P(V) \supset A(V)$. As is known, both indicated groups are Lie transformation groups. Let $P_0(V)$ and $A_0(V)$ be the connected components of the identity of the indicated groups. The present article is devoted mainly to the proof of the following theorem.

Theorem 1. *If a complete analytic Riemannian space V of dimension $n \geqslant 3$ is not a space of constant positive curvature, then the group $P_0(V)$ coincides with $A_0(V)$.*

2°. Let us recall some facts from the theory of projective transformations of Riemannian spaces ([1]). Suppose a projective mapping is given from the space V onto some other space $\bar{V}$. In any local coordinate system common to the mapping, the equalities hold

$$\bar{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) + \delta^i_j \theta_k(x) + \delta^i_k \theta_j(x),$$

where θ_k is the gradient of the function

$$\theta(x) = \frac{1}{2(n+1)} \ln \frac{\bar{\Delta}(x)}{\Delta(x)} \qquad (n = \dim V = \dim \bar{V}),$$

defined on the whole space V; here $\Delta(x)$ denotes the discriminant of the quadratic form determining the metric of the space V at the point x. In the case when the spaces V and $\bar{V}$ are analytic, the function θ is also analytic.

Let $G = \{g_t \mid -\infty < t < \infty\}$ be a one-parameter subgroup of the group $P(V)$. From what was said above it follows that

$$\Gamma^i_{jk}(x,t) = \Gamma^i_{jk}(x) + \delta^i_j \theta_k(x,t) + \delta^i_k \theta_j(x,t),$$

where $\Gamma^i_{jk}(x,t)$ are the coefficients of the "carried along" connection** at the point x. In this case

$$\theta(x,t) = \frac{1}{2(n+1)} \ln \frac{\Delta(x,t)}{\Delta(x)},$$

where $\Delta(x,t)$ is the discriminant of the carried-along metric at the point x. In what follows an important role is played by the function $\psi(x) = -\partial\theta(x,t)/\partial t|_{t=0}$. If the space V is analytic, then $\psi(x)$ is an analytic function defined on all of V.

To the subgroup G there corresponds a vector field

$$\xi(x) = \partial g_t(x)/\partial t|_{t=0} \qquad (x \in V)$$

on V. Consider the tensor field $\xi_{(i,j)} = \xi_{i,j} + \xi_{j,i}$. We shall call a point x **ordinary** with respect to the group G if the number of distinct roots

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$*$ A projective transformation of the space V is a diffeomorphism preserving the system of geodesic lines; an affine transformation is a diffeomorphism preserving the affine connection of the space V.

$**$ The transformation g_t induces in the space V a new, "carried-along" connection (at an arbitrary point $x \in V$ this connection is obtained by carrying the connection of the space V at the point $g_{-t}(x)$ by means of the transformation g_t). The quantities $\Gamma^i_{jk}(x)$ and $\Gamma^i_{jk}(x,t)$ are the coefficients of the original and the carried-along connections in one and the same local coordinate system.

$\lambda_1, \dots, \lambda_p$ of the equation $\det(\xi_{(i,j)} - \lambda g_{ij}) = 0$ are constant in some neighborhood of this point.

Proposition 1. Let $G \subset P(V)$, $G \not\subset A(V)$, and let x be ordinary for G. Then, in some local coordinate system in a neighborhood of this point,

$$ds^2 = d\omega_1^2 + d\omega_2^2 + \dots + d\omega_p^2, \tag{1}$$

$$\xi_{(i,j)}\, dx^i dx^j = \lambda_1 d\omega_1^2 + \lambda_2 d\omega_2^2 + \dots + \lambda_p d\omega_p^2, \tag{2}$$

where

$$d\omega_\alpha^2 = \prod_{\beta=1}^{p}{}' |f_\beta - f_\alpha|\, ds_\alpha^2, \qquad \lambda_\alpha = f_\alpha + \sum_{\beta=1}^{p} f_\beta \quad (\alpha = 1, 2, \dots, p);$$

moreover, the function ψ has the form

$$\psi = \frac{1}{2}\sum_{\beta=1}^{p} f_\beta. \tag{3}$$

Let us explain what has been said. In the notation (1) it is assumed that the coordinates $x^1, x^2, \dots, x^n$ are divided into $p > 1$ groups $(x^{i_1}), (x^{i_2}), \dots, (x^{i_p})$, and: 1) ds_α^2 is a positive definite metric form depending only on the coordinates x^{i_α}; 2) f_α is a function of x^{i_α}, if the group (x^{i_α}) consists of only one coordinate; $f_\alpha = \text{const}$, if the group (x^{i_α}) contains more than one coordinate; at least one of the functions f_α is nonconstant; 3)

$$\prod_{\beta=1}^{p}{}' |f_\beta - f_\alpha|$$

denotes the product of the differences $|f_\beta - f_\alpha|$ over all $\beta = 1, \dots, p$, except $\beta = \alpha$; $f_\alpha \neq f_\beta$ for $\alpha \neq \beta$.

If the space V is analytic, then the forms ds_α^2 and the functions f_α are also analytic.

Let, for definiteness, the nonconstant functions be $f_1, f_2, \dots, f_r$, and $f_{r+1} = \text{const}, \dots, f_p = \text{const}$ $(0 < r \leq p)$. The forms $ds_1^2, \dots, ds_r^2$ are, of course, one-dimensional; we shall take them equal to $(dx^1)^2, \dots, (dx^r)^2$ (respectively). Thus,

$$\begin{gathered} ds^2 = \Pi' |f_\beta - f_1| (dx^1)^2 + \dots + \Pi' |f_\beta - f_r| (dx^r)^2 + \\ + \Pi' |f_\beta - f_{r+1}| \, ds_{r+1}^2 + \dots + \Pi' |f_\beta - f_p| \, ds_p^2. \end{gathered} \tag{4}$$

The metric

$$\begin{gathered} {}^*ds^2 = \Pi' |f_\beta - f_1| (dx^1)^2 + \dots + \Pi' |f_\beta - f_r| (dx^r)^2 + \\ + \Pi' |f_\beta - f_{r+1}| (dy^{r+1})^2 + \dots + \Pi' |f_\beta - f_p| (dy^p)^2 \end{gathered}$$

(where the coordinates are the p variables $x^1, \dots, x^r, y^{r+1}, \dots, y^p$) is called the metric associated with (4).

Proposition 2. If ${}^*ds^2$ does not have constant curvature and $p \geq 3$, then the components $\xi_a^i(x)$ of the vector $\xi(x)$ $(a = 1, \dots, p)$ depend only on the coordinates x^i of the same group (i.e., on ds_a^2); in particular, $\xi^1 = \xi^1(x^1), \dots, \xi^r = \xi^r(x^r)$. Taking this fact into account, from (2) there follow the equations

$$\xi^i \, df_i / dx^i = f_i^2 - c f_i + a, \tag{5}$$

$$(i = 1, \dots, r),$$

$$2\,d\xi^i/dx^i = (3-p)f_i - c(p-1), \tag{6}$$

where a and c are constants. As for the remaining functions $f_{r+1}, \dots, f_p$, these must be roots of the equation $f^2 + cf + a = 0$ (in particular, the number $p-r$ of such functions does not exceed two).

3°. Proof of Theorem 1. Arguing by contradiction, suppose that $\dim P(V) > \dim A(V)$. Then there exists a one-parameter group G belonging to $P(V)$, but not contained in $A(V)$. From what was said above there follows the existence in the space V of a domain U and, in it, a local coordinate system in which (1), (2), and (3) hold. Two cases are possible.

Case 1. The associated metric $\overset{*}{ds^2}$ does not have constant curvature. Consider the trajectory of some point $x_0(x_0^1, \dots, x_0^n) \in U$ under the action of the group G: $x(t) = g_t(x_0)$. Along a segment of the trajectory lying in the domain U, each of the functions $f_i(x^i)$ $(i = 1, \dots, r)$ is transformed into a function $f_i^*(t)$, defined on some set of values of t. As for the function $\psi(x)$, defined on the whole space V, along the indicated trajectory it is transformed into a function $\psi^*(t)$, defined and analytic on the whole t-axis. By virtue of (5) we have

$$df_i^*/dt = f_i^{*2} + cf_i^* + a,$$

whence it follows that f_i^* is one of the functions

$$-c/2 + k\,\mathrm{tg}\,k(t+c_i) \qquad (\text{for } D<0); \tag{7}$$

$$-c/2 - k\,\mathrm{th}\,k(t+c_i) \tag{8}$$

or

$$-c/2 - k\,\mathrm{cth}\,k(t+c_i) \qquad (\text{for } D>0); \tag{9}$$

$$-c/2 - 1/(t+c_i) \qquad (\text{for } D=0), \tag{10}$$

where $D = c^2/4 - a$ and $k = |D|^{1/2}$. But for all values of t from some interval we have

$$\psi^*(t) = \frac{1}{2}\sum_1^p f_\beta^*(t).$$

Hence, from the condition that $\psi^*(t)$ is analytic on the whole t-axis, it follows that the cases (7), (9), and (10) are impossible, i.e., necessarily one must have $D > 0$ and $f_i^*(t) = -c/2 - k\,\mathrm{th}\,k(t + c_i)$. In particular,

$$\mu_1 < f_i(x_0^i) < \mu_2 \qquad (i = 1, \dots, r), \tag{11}$$

where μ_1 and μ_2 are the roots of the equation $f^2 + cf + a = 0$. Changing, if necessary, the numbering of the coordinates $x^1, x^2, \dots, x^r$, one may assume that the inequalities $f_1(x_0^1) < f_2(x_0^2) < \cdots < f_r(x_0^r)$ hold. Thus,

$$\mu_1 < f_1(x_0^1) < f_2(x_0^2) < \cdots < f_r(x_0^r) < \mu_2.$$

Eliminating the function ξ^i from equations (5), (6), and taking into account the condition $D > 0$, it is easy to obtain, for the functions $f_1(x^1), \dots, f_r(x^r)$, the differential equations

$$df_i/dx^i = A_i\big(|f_i - \mu_2|^{-\mu_1}|f_i - \mu_1|^{\mu_2}\big)^{(p+1)/4D} \qquad (A_i = \mathrm{const} \neq 0). \tag{12}$$

For any solution of such an equation satisfying the initial condition (11), there exists an interval of monotone increase from μ_1 to μ_2 (if $A_i > 0$) or an interval of monotone decrease from μ_2 to μ_1 (if $A_i < 0$). Using this, one may, starting from x_0^1, by a monotone change of the coordinate x^1 arrange that the value of the function $f_1(x^1)$ (or of its analytic continuation), increasing, comes arbitrarily close to the value $f_2(x_0^2)$. Let, for example, $A_1 > 0$. Consider the arithmetic n-dimensional space R^n and in it the curve γ: $x^1 = t$, $x^2 = x_0^2, \dots, x^n = x_0^n$, corresponding to the indicated change of the coordinate x^1 ($x_0^1 \le t < x_{00}^1$, $f_1(x_{00}^1) = f_2(x_0^2)$). In some neighborhood of the curve γ in R^n the quadratic form (4) remains positive definite and analytic. Thereby some neighborhood of γ is transformed into an analytic Riemannian space. Denote the latter by V'; one may assume that V' is simply connected. The length of the curve γ in V' is obviously finite. The spaces V' and V are Riemannian analytic, and moreover some neighborhoods of the points $(x_0^1, \dots, x_0^n) \in V'$ and $x_0 \in V$ are isometric. Since V' is simply connected and V is complete, the indicated isometry extends to a (single-valued) locally isometric mapping of V' into V. Thus, V' (and consequently also γ) may be regarded as immersed in V. But in a neighborhood of the point $x_0 \in V$ we have

$$|\,\mathrm{grad}\,\psi|^2 = g^{ij}\psi_i\psi_j = \sum_{i=1}^{r} \frac{f_{i'}^2}{g_{ii}},$$

and under the above-indicated change of the coordinate x^1 this expression will tend to infinity (for $g_{11} \to 0$ and $g_{22} \to 0$), whereas the limit $\cdots$

values for f_1' and f_2' are nonzero by virtue of (12)). This contradicts the completeness of the space V, and also the fact that the function ψ is defined and analytic on all of V. Thus, assuming that $d\dot{s}^2$ does not have constant curvature, we arrive at a contradiction—case 1 is impossible.

Case 2. The metric $d\dot{s}^2$ has constant curvature K. Put

$$F_\alpha = \prod_{\beta=1}^{p} |f_\beta - f_\alpha|, \qquad ds_0^2 = F_1 (dx^1)^2 + \dots + F_r (dx^r)^2,$$

then ds^2 can be written in the "semi-reduced" form

$$ds^2 = ds_0^2 + F_{r+1} ds_{r+1}^2 + \dots + F_{pds} p^2$$

(the functions $F_{r+1}, \dots, F_p$ depend on the variables from ds_0^2); here the associated metric

$$d\dot{s}^2 = ds_0^2 + F_{r+1} (dy^{r+1})^2 + \dots + F_p (dy^p)^2$$

has constant curvature K.

Let $K \geq 0$. Consider an r-dimensional totally geodesic leaf carrying the metric ds_0^2. The maximal analytic continuation (see ([2])) of this leaf in the space V is a complete Riemannian space T; its universal covering $\hat{T}$ in the case $K = 0$ is the Euclidean space E^r, and in the case $K > 0$ is the sphere S^r. From the expression for the functions $F_{r+1}, \dots, F_p$ (([1]), §9) it is easy to see that, under analytic continuation to the whole space $\hat{T}$, each of them vanishes somewhere.* Hence it follows ([2]) that each of the metrics $ds_{r+1}^2, \dots, ds_p^2$ has constant positive curvature, and in such a way that the entire metric ds^2 has constant curvature K.

Let now $K < 0$. We shall use the following obvious proposition. If M and $\bar{M}$ are two complete Riemannian spaces of constant negative curvature and φ is a projective mapping of one of them onto the other, then φ is a homothety (and, when the curvatures are equal, an isometry).

Consider, as in the case $K \geq 0$, a totally geodesic submanifold T of the space V. The restriction $g_t|_T$ is a projective mapping of the submanifold T onto some other submanifold $g_t(T)$. Since both spaces T and $g_t(T)$ have constant negative curvature, the mapping $g_t|_T$ is a homothety. It follows easily that the function ψ associated with the given group G is constant, and this means that $G \subset A(V)$. The theorem is completely proved.

4°. As shown in ([1]), a metric of the form (4) is reducible only in the case when the associated metric has constant curvature zero. But, as was shown above, in this case the metric (4) itself has constant curvature zero, or else all the functions

$f_1, \dots, f_p$ are constant. Hence, by the well-known theorem of Levi-Civita (see, for example, ([1])), it follows that the space V (complete analytic) admitting a non-affine projective transformation is locally irreducible.

On the other hand, it is not difficult to show that in a locally irreducible complete space V the group $A(V)$ coincides with $I(V)$—the group of all isometric transformations of the space V. Hence the following addition to the theorem proved above follows.

Let V be complete and analytic. If V is locally reducible, then the group $P(V)$ coincides with $A(V)$. If V is locally irreducible and does not have constant positive curvature, then the group $P_0(V)$ coincides with $I_0(V)$.

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REFERENCES

[1] A. S. Solodovnikov, *Uspekhi Mat. Nauk*, **11**, no. 4 (70) (1956).
[2] A. S. Solodovnikov, *Dokl. Akad. Nauk SSSR*, **177**, no. 3 (1967).

* Note that among the indicated functions there are no constants; this follows from the fact that at least one of the functions $f_1, \dots, f_p$ is nonconstant.

Note: Figure translations are in progress. See original paper for figures.