ESTIMATES OF THE GROWTH OF DERIVATIVES OF SOLUTIONS OF HOMOGENEOUS ELLIPTIC EQUATIONS NEAR THE BOUNDARY

1969

SovietRxiv

Abstract

Full Text

UDC 517.944

Yu. P. KRASOVSKII

ESTIMATES OF THE GROWTH OF DERIVATIVES OF SOLUTIONS OF HOMOGENEOUS ELLIPTIC EQUATIONS NEAR THE BOUNDARY

(Presented by Academician S. L. Sobolev on May 13, 1968)

A solution of a homogeneous elliptic equation that is smooth inside a domain may grow as it approaches the boundary. We determine the exact order of growth of the solution and of its derivatives as a function of the differential properties of the right-hand sides of the boundary-value problem. In addition, we obtain estimates that generalize the well-known maximum principle to general boundary-value problems. All the results listed below will be obtained on the basis of estimates for derivatives of Green' s functions established in ([1]).

Consider the boundary-value problem

$$\mathcal{L}u = 0 \quad \text{for } x \in \Omega; \qquad B_j u = \varphi_j(x) \quad \text{for } x \in S. \tag{1}$$

Here $j = 1, \dots, m$; Ω is a closed bounded domain of n-dimensional space ($n \geq 2$); $x = (x_1, \dots, x_n)$; S is the boundary of Ω; $\mathcal{L}$ is an elliptic operator of order $2m$, defined in Ω; B_j are differential operators of order $m_j \leq 2m - 1$, defined on S:

$$\mathcal{L} \equiv \sum_{0 \leq |\beta| \leq 2m} a_\beta(x) D_x^\beta, \qquad B_j \equiv \sum_{0 \leq |\beta| \leq m_j} b_{j\beta}(x) D_x^\beta \left(D_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right).$$

Our assumptions concerning the operators $\mathcal{L}$, B_j and the boundary S are as follows: $\mathcal{L}$ is properly elliptic; the B_j cover $\mathcal{L}$; all coefficients $a_\beta(x) \in C^{l_1+1}(\Omega)$, $b_{j\beta}(x) \in C^{l_1+1}(S)$; the boundary $S \in C^{l_1+2m+1}$ in the sense of ([2]); l_1 is an integer satisfying the condition $l_1 \geq \max(2l_0 - 1, l_0 + 1)$, where $l_0 = \max_j (2m - m_j)$.

Let $x_i \in S$ be an arbitrary point; $x' = V(x)$ an invertible change of coordinates that straightens the boundary in a neighborhood of x_i. The transformation $x' = V(x)$ maps the part $S - S_{x_i d}$ onto a piece of the plane $x'_n = 0$, $|\bar{x}'| \leq d$; $\bar{x}' = (x'_1, \dots, x'_{n-1})$; $d > 0$ is a constant depending on S.

Let $r \geq 0$ be an integer satisfying the condition $r < \max_j m_j$.

We shall assume that for those j for which $m_j > r$, the function $\varphi_j(x)$ is representable in the form

$$\varphi_j(V^{-1}(x')) = \sum_{0 \leq |\beta| \leq m_j - r} D^{\beta}_{x'} \Phi^{(i)}_{j\beta}(\bar{x}'), * \tag{2}$$

where the functions $\Phi^{(i)}_{j\beta}(\bar{x}')$ are defined for $|\bar{x}'| \leq d$ and have derivatives $D^{\beta}_{x'} \Phi^{(i)}_{j\beta}(\bar{x}')$. The representation (2) holds in each neighborhood $S_{x_i d}$, where $S_{x_i\, d/2}$ $(i = 1, \dots, N_0)$ is a system of neighborhoods covering S.

Define the constants:

$$K_j = \max_i \sum_{0 \leq |\beta| \leq m_j - r} \left| \Phi^{(i)}_{j\beta} \right|_{|\bar{x}'| \leq d},$$

$$K_{j\alpha} = \max_i \sum_{0 \leq |\beta| \leq m_j - r} \left| \Phi^{(i)}_{j\beta} \right|^{|\bar{x}'| \leq d}_{\alpha}.$$

Here the maximum is taken over all $i = 1, \dots, N_0$; $|\cdot|_{k,D}$, $|\cdot|_{k+\alpha,D}$ are norms in the spaces $C^k(D)$ and $C^{k+\alpha}(D)$ (see, for example, ([2])); $k \geq 0$ is an integer; $0 < \alpha < 1$.

In what follows, by $\rho(x)$ we denote the distance from $x \in \Omega$ to S; $|u|_{L_1}$ is the integral of $|u(x)|$ over Ω.

Theorem 1. Let $u(x) \in C^{2m+l_1-l_0+\alpha}(\Omega)$ be a solution of problem (1); $r \geq 0$ an integer; $l_1 - 2m + 1 + r \geq 0$.

Then any derivative $D^t_x u(x)$, for x lying inside Ω, $t \leq 2m + l_1 - l_0$, admits the estimates:

$$|D^t_x u(x)| \leq M\{[\rho(x)]^{r-t+\alpha} + 1\} \left(\sum_j {}_1 K_{j\alpha} + \sum_j {}_2 |\varphi_j|^S_{r-m_j+\alpha} + |u|_{L_1} \right), \tag{3}$$

$$|D^t_x u(x)| \leq M\{[\rho(x)]^{r-t} + |\ln \rho(x)| + 1\} \left(\sum_j {}_1 K_j + \sum_j {}_2 |\varphi_j|^S_{r-m_j} + |u|_{L_1} \right), \tag{4}$$

where $\sum_1$ extends over those j for which $m_j > r$, and $\sum_2$ over those j for which $m_j \leq r$; $\ln \rho(x)$ may be omitted if $r - t \neq 0$; the constant M (here and below) depends only on the coefficients of $\mathcal{L}$, B_j, the boundary S, and the numbers r, t, n, m, α.

The proof is based on the following representation of the solution $u(x)$, obtained in (1):

$$u(x) = \sum_{j=1}^{m} \int_S \mathcal{G}_j^{(N)}(x,y)\varphi_j(y)\,dy + v_N(x) = u_N(x) + v_N(x), \qquad N = 2l_1 + n + 3.$$

Here $\mathcal{G}_j^{(N)}(x,y)$ are the principal parts of the Green functions of problem (1), and $u_N(x)$ is the principal part of the solution.

Consider the function

$$u_{N_j}(x) = \int_S \mathcal{G}_j^{(N)}(x,y)\varphi_j(y)\,dy. \tag{5}$$

Let j be such that $m_j > r$. In order to obtain an estimate of $D_x^k u_{N_j}(x)$ for x lying inside Ω in a neighborhood of the point $x_i \in S$, it suffices for us to obtain an estimate of the function

$$u'_{N_j}(x) = \int_S \mathcal{G}_j^{(N)}(x,y)\chi(y)\varphi_j(y)\,dy,$$

where $\chi(y)$ is an everywhere infinitely differentiable function equal to 0 for $|y - x_i| \geq d$; $\chi(y) = 1$ for $|y - x_i| \leq d/2$.

After the change of variables $x' = V(x)$, $y' = V(y)$, taking (2) into account, we obtain

$$u'_{N_j}(V^{-1}(x')) = \sum_{0 \leq |\beta| \leq m_j - r} v_\beta(x'), \tag{6}$$

$$v_\beta(x') = (-1)^{|\beta|} \int_{|\bar{y}'| \leq d} D_{\bar{y}'}^{\beta} \left\{ [\mathcal{G}_j^{(N)}(x,y)\chi(y)]_{\substack{x = V^{-1}(x') \\ y = V^{-1}(\bar{y}')}} D(\bar{y}') \right\} \Phi_{j\beta}(\bar{y}')\,d\bar{y}'.$$

Here $d\bar{y}'$ is the area element of the plane $y'_n = 0$, $dy = D(\bar{y}')\,d\bar{y}'$.

If $|\beta| = m_j - r$, then we write $D_{x'}^k v_\beta(x')$ in the form

$$D_{x'}^k v_\beta(x') = (-1)^{|\beta|} \int_{|\bar{y}'| \leq d} D_{x'}^k D_{\bar{y}'}^{\beta} \left\{ [\mathcal{G}_j^{(N)}(x,y)\chi(y)]_{\substack{x = V^{-1}(x') \\ y = V^{-1}(\bar{y}')}} \times D(\bar{y}') \right\} [\Phi_{j\beta}(\bar{y}') - \Phi_{j\beta}(\bar{x}')]\,d\bar{y}'. \tag{7}$$

Now, taking into account the estimates for the derivatives of $\mathcal{G}_j^{(N)}(x,y)$ (1):

$$|D_x^t D_y^s \mathcal{G}_j^{(N)}(x,y)| \leq M[|x-y|^\lambda + |\ln|x-y|| + 1], \tag{8}$$

where $\lambda = m_j - n + 1 - t - s$, and $\ln|x-y|$ may be omitted when $\lambda \neq 0$, from (7) we obtain

$$|D_{x'}^k v_\beta(x')| \leq M'[(x'_n)^{r-k+\alpha} + 1]K_{j\alpha}. \tag{9}$$

Here D_x^k is any derivative of order k; x' lies inside the half-ball $|x'| \leq d/2$; $x'_n \geq 0$. The terms entering into (6), for $|\beta| < m_j - r$, are estimated directly with the aid of (8), and, consequently, estimate (9) is valid for any β. Therefore, on the basis of (6), (9) we obtain

$$\left|D_x^k u_{N_j}(x)\right| \le M\left\{[\rho(x)]^{r-k+\alpha}+1\right\} K_{j\alpha}. \tag{10}$$

Here x is any point inside Ω.

Let now j be such that $m_j \le r$. In this case we write $D_x^k u_{N_j}(x)$, for x lying inside Ω, in the form

$$D_x^k u_{N_j}(x) = \int_S D_x^k \mathcal{G}_j^{(N)}(x,y)[\varphi_j(y)-P_t(y)]\,dy+$$

$$+\int_S D_x^k \mathcal{G}_j^{(N)}(x,y)P_t(y)\,dy = v'(x)+v''(x), \tag{11}$$

where $P_t(y)$ is the Taylor polynomial of order $t=r-m_j$ of the function $\varphi_j(y)$ at the point $x^* \in S$—the point of S nearest to x, so that the estimate holds:

$$|\varphi_j(y)-P_t(y)| \le M|\varphi_j|^S_{r-m_j+\alpha}|x-y|^{r-m_j+\alpha}. \tag{12}$$

With the aid of this estimate and estimates (8) we obtain:

$$|v'(x)| \le M'\left\{[\rho(x)]^{r-k+\alpha}+1\right\}|\varphi_j|^S_{r-m_j+\alpha}.$$

An analogous estimate for $v''(x)$ follows from the properties of the operator $\mathcal{G}_j^{(N)}$ (see ([1])). Therefore we have:

$$\left|D_x^k u_{N_j}(x)\right| \le M\left\{[\rho(x)]^{r-k+\alpha}+1\right\}|\varphi_j|^S_{r-m_j+\alpha}. \tag{13}$$

We have obtained estimates of $D_x^k u_{N_j}(x)$ in terms of the Hölder norms of the right-hand sides. Estimates of $D_x^k u_{N_j}(x)$ in terms of K_j, $|\varphi_j|^S_{r-m_j}$ are derived analogously, with the only difference that in (11) the difference $\Phi_{j\beta}(\bar{y}')-\Phi_{j\beta}(\bar{x}')$ is replaced by $\Phi_{j\beta}(\bar{y}')$, and instead of estimate (12) the estimate is used:

$$|\varphi_j(y)-P_t(y)| \le M|\varphi_j|^S_{r-m_j}|x-y|^{r-m_j};$$

for $t=r-m_j-1 \ge 0$; if $r-m_j=0$, then in (11) one should put $P_t(y)=0$.

For the function $v_N(x)$ the estimate is valid

$$|D_x^k v_N(x)| \leq M \left(\sum_j^1 K_j + \sum_j^2 |\varphi_j|_{r-m_j}^S + |u|_{L_1} \right),$$

which follows from the definition of this function as the solution of an elliptic boundary value problem with obviously smooth right-hand sides (see ([1])). The theorem is proved.

Theorem 2. Let $u(x) \in C^{2m-\alpha}(\Omega)$ be a solution of problem (1), $l_1 - 2m + 1 + r \geq 0$, r an integer, $0 \leq r \leq 2m$.

Then the estimate holds

$$|u|_{r+\alpha}^{\Omega} \leq M \left(\sum_j^1 K_{j\alpha} + \sum_j^2 |\varphi_j|_{r-m_j+\alpha}^S + |u|_{L_1} \right). \tag{14}$$

To derive this estimate it is enough to use estimate (3) for $t \leq r+1$ and one estimate from ([2]), p. 175.

Let us note that estimate (14), for $r \geq m'$ (m' is the maximal order of differentiation in the direction of the normal to S occurring in the operators B_j $(j = 1, \dots, m)$), was obtained in ([2]). Next we shall obtain estimates that may be called a generalization of the maximum principle.

Let $D_{x'}^k$ be any derivative of order $k \geq 0$ with respect to the variables $x'_1, \dots, x'_{n-1}$, where $x'_1, \dots, x'_n$ is a local coordinate system in a neighborhood of some point $x_0 \in S$; $x'_n = 0$ is the equation of S in a neighborhood of x_0. We shall regard the derivative $D_{x'}^k$ as defined in the boundary strip of the domain Ω $(\rho(x) \leq d)$.

Introduce the operators $\widetilde{B}_j$, defining them as follows:

$$\widetilde{B}_j = \sum_{0 \leq |\beta| \leq m_j} \tilde{b}_{j\beta}(x) D_x^\beta.$$

Here $\tilde{b}_{j\beta}(x)$ are functions defined in Ω, and $\tilde{b}_{j\beta}(x) = b_{j\beta}(x)$ for $x \in S$; all $\tilde{b}_{j\beta}(x) \in C^{l+1}(\Omega)$. The operators $\widetilde{B}_j$ are defined in Ω and coincide on S with the operators B_j.

Lemma 1. *The following estimates are valid:*

$$\left| D_y^t D_x^k \widetilde{B}_i \mathcal{G}_j^{(N)}(x,y) \right| \leq M \left\{ [\rho(x)]^\alpha |x-y|^{\lambda-\alpha} + |x-y|^{\lambda+\alpha} + 1 \right\}, \tag{15}$$

where $\lambda = m_j - m_i - k - t - n + 1,\ t \leq l_1,\ k \leq l_1 - l_0 + 2m - m_i$ $(i, j = 1, \dots, m)$.

For the proof, write $D_y^t D_x^k \widetilde{B}_i \mathcal{G}_j^{(N)}$ in the form

$$D_y^t D_x^k \widetilde{B}_i \mathcal{G}_j^{(N)}(x,y) = \left[D_y^t D_x^k B_i \mathcal{G}_j^{(N)}(x,y)\right]_{x=x^*} +$$

$$+ \left\{D_y^t D_x^k \widetilde{B}_i \mathcal{G}_j^{(N)}(x,y) - \left[D_y^t D_x^k B_j \mathcal{G}_j^{(N)}(x,y)\right]_{x=x^*}\right\}, \tag{16}$$

where $x^* \in S$, $|x^* - y| = |x - y|$. Note that the first term in (16) is a smooth function of x^*, y, since $B_i \mathcal{G}_j^{(N)}(x,y) = h_{ij}^{(N)}(x,y)$ (see (1)). Therefore, using, to estimate the derivatives of the same name entering the braces, Lagrange' s finite-increment formula, with the aid of (8) we obtain (15).

Theorem 3. *Let $u(x) \in C^{2m+l_1-l_0}(\Omega)$ be a solution of problem (1); $m_i \le r \le 2m + l_1 - l_0$, r an integer.*

Then the function $D_\xi^{r-m_i} \widetilde{B}_i u(x)$ in the boundary strip Ω $(\rho(x) \le d)$ admits the estimate

$$\left|D_x^{r-m_i} \widetilde{B}_i u(x)\right| \le M \left(\sum_j {}_1 K_j + \sum_j {}_2 |\varphi_j|_{r-m_j}^S + |u|_{L_1}\right), \tag{17}$$

where $\sum_1$, $\sum_2$ are the same as in Theorem 1.

The proof is carried out according to the same scheme as in deriving estimate (4). One need only write everywhere, instead of D_x^k, $D_x^{r-m_i} \widetilde{B}_i$, and, when considering the integrals (7), (11), use the estimates (15).

In the case of the Dirichlet problem ($B_j \equiv \partial^{j-1}/\partial\nu^{j-1}$; $\partial/\partial\nu$ is the derivative along the normal to S, $j = 1, \ldots, m$), from (17) there follows the estimate obtained in (3):

$$|u|_{m-1}^\Omega \le M \left(\sum_{j=1}^m |\varphi_t|_{m-j}^S + |u|_{L_1}\right).$$

We note that the estimates of Theorems 1-3 remain valid also for generalized solutions of problem (1), if the $\varphi_j(x)$ in (1) are regarded as generalized functions.

Rostov-on-Don Institute
of Agricultural Machine Building

Received
11 V 1968

REFERENCES

1. Yu. P. Krasovskii, *Izv. AN SSSR, ser. matem.*, **31**, 977 (1967).

2. S. Agmon, A. Douglis, D. Nirenberg, *Estimates of solutions of elliptic equations near the boundary*, IL, 1962.

3. Spanne Sven, *C. R.*, **262**, No. 25, A1407–A1410 (1966).

Note: Figure translations are in progress. See original paper for figures.