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COMPUTATION OF REFLECTION AND TRANSMISSION COEFFICIENTS OF ELECTROMAGNETIC WAVES FOR A SPHERICALLY LAYERED ANISOTROPIC LAYER BY THE METHOD OF RECALCULATION OF IMPEDANCE AND FIELDS

MATHEMATICAL PHYSICS

Abstract

Full Text

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COMPUTATION OF REFLECTION AND TRANSMISSION COEFFICIENTS OF ELECTROMAGNETIC WAVES FOR A SPHERICALLY LAYERED ANISOTROPIC LAYER BY THE METHOD OF RECALCULATION OF IMPEDANCE AND FIELDS

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1. In paper ([1]) it was shown how, for a spherically layered anisotropic medium specified by the tensor of dielectric constant

$$\varepsilon(r) = \begin{pmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{r\varphi} \\ \varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta\varphi} \\ \varepsilon_{\varphi r} & \varepsilon_{\varphi\theta} & \varepsilon_{\varphi\varphi} \end{pmatrix}, \tag{1}$$

one can determine the parameters of the normal waves with high accuracy. This was done on an electronic computer by means of the method of recalculation of the 2×2 impedance matrix $\mathfrak{Z}$. Now, assuming that the indicated medium is inhomogeneous only in a layer of finite thickness (for $r \leq l$, $\varepsilon = 1$, and as $r \to \infty$, $\varepsilon \to 1$), we shall apply this method to computing the reflection coefficients R from such a layer and the transmission coefficients T of a wave through it:

$$R = \begin{Vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{Vmatrix}, \qquad T = \begin{Vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{Vmatrix}. \tag{2}$$

The coefficients of the upper rows of (2) refer to the vertically polarized wave, and the coefficients of the lower rows of (2) to the horizontally polarized wave incident on the layer ($l < r < r_\infty$) from above.

2. To compute R it is sufficient to know the impedance $\mathfrak{Z}_0 = \mathfrak{Z}(r_0)$ at the point $r_0 < l$, recalculated from the point r_∞, located above the layer, where ε is sufficiently close to 1, and $\mathfrak{Z} = \mathfrak{Z}_\infty = \mathfrak{Z}(r_\infty)$. According to (5) ([1]), this recalculation is performed by means of a system of 4 nonlinear first-order differential equations of Riccati type.

To derive the formula relating the matrix R to the matrix $\mathfrak{Z}_0$, let us write the asymptotic expressions of the solutions of the wave equation in the region $r < l$ for vertical and horizontal polarizations in the form of a sum of incident and reflected waves:

$$\sqrt{\sin\theta}\, Be^{-i\omega t} \simeq \{ae^{i[\nu\theta+\bar{k}r]} + ce^{i[\nu\theta-\bar{k}r]}\}e^{-i\omega t},$$

$$\sqrt{\sin\theta}\, Ae^{-i\omega t} \simeq \{be^{i[\nu\theta+\bar{k}r]} + de^{i[\nu\theta-\bar{k}r]}\}e^{-i\omega t}. \tag{3}$$

Here B and A are scalar potentials through which the field amplitudes are expressed,

$$\bar{H}_\varphi = rH_\varphi = \frac{\partial B}{\partial\theta}, \qquad \bar{E}_\varphi = rE_\varphi = \frac{\partial A}{\partial\theta}, \qquad \bar{E}_\theta = -\frac{i}{k}\frac{\partial^2 B}{\partial r\,\partial\theta},$$

$$\bar{H}_\theta = \frac{i}{k}\frac{\partial^2 A}{\partial r\,\partial\theta}; \tag{3'}$$

a, b are the amplitudes of the incident waves, c, d are the amplitudes of the reflected waves; $\bar{k} = k\cos\psi$; k is the wave number of the free medium; ψ is the angle of incidence of the wave on the layer. Similarly, the waves that have passed through the layer, as $r \to \infty$, can be

written in the form

$$\sqrt{\sin\theta}\, Be^{-i\omega t} = fe^{i[\nu\theta+kr]}e^{-i\omega t}, \qquad \sqrt{\sin\theta}\, Ae^{-i\omega t} = ge^{i[\nu\theta+kr]}e^{-i\omega t}. \tag{4}$$

If new wave amplitudes are introduced for the field components $\bar{a} = i\nu a$, $\bar{c} = i\nu c$, ..., then the relation between the reflection and transmission coefficients and the wave amplitudes will have the form

$$\begin{pmatrix}\bar{c}\\ \bar{d}\end{pmatrix} = \begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}\begin{pmatrix}\bar{a}\\ \bar{b}\end{pmatrix}, \qquad \begin{pmatrix} f\\ g\end{pmatrix} = \begin{pmatrix} t_{11} & t_{12}\\ t_{21} & t_{22}\end{pmatrix}\begin{pmatrix}\bar{a}\\ \bar{b}\end{pmatrix}. \tag{5}$$

Using (3) and (3′), we obtain for the field components the expressions

$$\bar{H}_\varphi = (\bar{a}+\bar{c}), \qquad \bar{E}_\theta = \cos\psi\,(\bar{a}-\bar{c}), \qquad \bar{E}_\varphi = \bar{b}+\bar{d}, \qquad \bar{H}_\theta = -\cos\psi\,(\bar{b}-\bar{d}). \tag{6}$$

On the other hand, by the definition of the impedance matrix we have

$$\bar{E}_\theta = z_{11}\bar{H}_\theta + z_{12}\bar{H}_\varphi, \qquad \bar{E}_\varphi = z_{21}\bar{H}_\theta + z_{22}\bar{H}_\varphi. \tag{7}$$

Eliminating the field components and wave amplitudes from (5), (6), and (7), we obtain the expression for the relation between $\mathfrak{Z}$ and R:

$$R = \frac{1}{\Delta} \begin{vmatrix} -z_{11}z_{22} + (\cos^{-1}\psi z_{12} - 1)(\cos\psi z_{21} - 1); & 2z_{11} \\ 2z_{22}; & z_{11}z_{22} - (\cos^{-1}\psi z_{12} + 1)(\cos\psi z_{21} + 1) \end{vmatrix}, \tag{8}$$

where

$$\Delta = z_{11}z_{22} - (\cos^{-1}\psi z_{12} + 1)(\cos\psi z_{21} - 1). \tag{8'}$$

The required values of the coefficients r_{ik} are obtained by substituting the values z_{ik} of the matrix $\mathfrak{Z}_0$ into expression (8).

3. To compute the matrix of transmission coefficients T, it is necessary to determine the electromagnetic fields $\bar{H}_\theta$ and $\bar{H}_\varphi$ that have penetrated through the layer. For this purpose we return to the original system of four linear differential equations for the field components (4) from (1), $\bar{E}_\theta, \bar{E}_\varphi, H_\theta$, and $\bar{H}_\varphi$, represented by the 4-vector $\bar{e}$:

$$\varepsilon_{rr}\frac{d\bar{e}}{dr} = ikA\bar{e}, \tag{9}$$

where A is a 4×4 matrix with elements composed of the elements of the tensor $\varepsilon(r)$ (see formula (4) of [1]). Using (7), we obtain from (9) a system of two linear differential equations for $\bar{H}_\theta$ and $\bar{H}_\varphi$

$$\begin{aligned} \varepsilon_{rr}\frac{d\bar{H}_\theta}{dr} &= ik[(-\bar{\varepsilon}^* z_{11} + \varepsilon_{rr}S^2 z_{21} - \delta z_{21})\bar{H}_\theta \\ &\quad + (-\bar{\varepsilon}^* z_{12} + \varepsilon_{rr}S^2 z_{22} - \delta z_{22} + S\varepsilon_{\varphi r})\bar{H}_\varphi], \\ \varepsilon_{rr}\frac{d\bar{H}_\varphi}{dr} &= ik[(\Delta z_{11} - \varepsilon^* z_{21})\bar{H}_\theta + (\Delta z_{12} - \varepsilon^* z_{22} - S\varepsilon_{\theta r})\bar{H}_\varphi], \end{aligned} \tag{10}$$

where $S = \sin\psi$, and ε^*, $\bar{\varepsilon}^*$, δ, and Δ are designations for combinations of the elements of the tensor ε, introduced in [1]. These equations should be integrated from $r = r_0$ to $r = r_\infty$ (i.e., in the direction opposite to the integration of the equations for $\mathfrak{Z}$) with the initial conditions

$$\bar{H}_\varphi = (1 + r_{11})a, \qquad \bar{H}_\theta = \cos\psi\, r_{21}a \tag{11'}$$

for an incident wave of vertical polarization and with the initial conditions

$$\bar{H}_\varphi = r_{12}b, \qquad \bar{H}_\theta = \cos\psi\,(r_{22} - 1)b \tag{11''}$$

Fig. 1

Figure 1: Fig. 1

Fig. 2

Figure 2: Fig. 2

Fig. 1

for an incident wave of horizontal polarization. Here the functions $z_{ik}(r)$ in equations (10) are obtained as the result of integrating the 3rd equation (5) from (1) for 3 from r_∞ to $r = r_0$. After integrating (10), for $r = r_\infty$ we obtain four quantities $\bar{H}_\varphi^{\mathrm{v}}, \bar{H}_\theta^{\mathrm{v}}, \bar{H}_\varphi^{\mathrm{g}}$, and $\bar{H}_\theta^{\mathrm{g}}$ (the index "v" refers to conditions (11′), and "g" to conditions (11″)). With their aid we determine the desired coefficients

$$\begin{aligned} t_{11} &= \bar{H}_\varphi^{\mathrm{v}}/a, \qquad t_{12} = \bar{H}_\theta^{\mathrm{v}}/a, \\ t_{21} &= \bar{H}_\varphi^{\mathrm{g}}/b, \qquad t_{22} = \bar{H}_\theta^{\mathrm{g}}/b. \end{aligned} \tag{12}$$

Fig. 2

4. The recalculation of impedances by equation (5) from (1) and the recalculation of fields by equation (10) are stable with respect to perturbations in the initial conditions, in contrast to field recalculations using the system of equations (9). As is known, the instability of integration in recalculating fields along the r axis of these equations has until now been a stumbling block for machine solutions of boundary-value problems in the theory of wave propagation in layered media. English authors, in particular Pitteway, in order to eliminate the instability when solving an equation analogous to (9), used the method of reorthogonalization ([2]). This laborious device, as follows from the foregoing, can be replaced by the more effective method of recalculating fields with the aid of equations (10), which we shall call the impedance method. The impedance method of field recalculation also makes it possible to compute the radial distributions of fields in normal waves.

Below are given the results of recalculations of impedance and fields for $\varepsilon(h)$, determined by the electron concentration profile $N_e(h)$, the effective collision-frequency profile $\nu_{\mathrm{eff}}(h)$, shown in Fig. 1, and the Earth's magnetic field H_0 with components $H_r = -0.443$, $H_\varphi = 0.169$, $H_\theta = -0.065$. They refer to the lower ionosphere at middle latitudes during summer noon on the GBR–Cambridge (England) path. In Fig. 2 are given hodographs $z_{ik}(h)$, beginning from the height $h_\infty = 95$ km for the wave of station GBR with frequency $f = 16$ kHz, $\psi = 30^\circ$.* The quantities r_{ik}, computed by formula (8), for the obtained z_{ik} at $h = 0$ will be equal to: $r_{11} = -0.0937 + i\,0.107$, $r_{12} = 0.0581 + i\,0.0337$, $r_{21} = 0.0780 + i\,0.0314$, $r_{22} = 0.00120 - i\,0.166$; their moduli are respectively equal to 0.142; 0.0672; 0.0842; 0.166 km.

Fig. 3

Figure 3: Fig. 3

Fig. 4

Figure 4: Fig. 4

In Figs. 3 and 4 are given hodographs of the fields $H_\varphi(h) = H'_\varphi + iH''_\varphi$ and $H_\theta(h) = H'_\theta + iH''_\theta$ for a vertically polarized wave incident at

* The numbers on the curves of Figs. 2 and 4 indicate h in kilometers.

at an angle of 30° (initial conditions (11)). The hodograph $H_\varphi(h)$ for $h < 60$ km represents the total field of two waves: an incident wave with amplitude equal to 1, and a reflected wave with amplitude equal to r_{11}. Therefore it has the form of an ellipse with semiaxes A and B, where $A - B = 2|r_{11}|$. In the present case $A \cong 1.15$, and $B \cong 0.85$, whence $|r_{11}| \cong 0.15$, which is close to the value of $|r_{11}|$ obtained from the data z_{ik} in Fig. 2 by formula (8). The inclination of the ellipse determines the phase of the reflected wave. One reduced wavelength

$$\bar{\lambda} = \lambda_0 / \cos\psi = 23 \text{ km}$$

fits along the arc of the ellipse. The point of the hodograph H_φ for $h < 60$ km moves counterclockwise, since the upward-going wave dominates. From 60 to 80 km, reflection of the waves and transformation of their polarizations occur. For $h > 80$ km, only one extraordinary wave remains, circularly polarized. It is represented by the hodographs $H_\varphi(h)$ and $H_\theta(h)$ in the form of circles of radii equal to 0.14, with the representing points moving with a phase shift of 90°. From Figs. 3 and 4, according to (12), it follows that the transmission coefficients $t_{11} \cong 0.14$, and $t_{21} \cong i\,0.14$. One wavelength of the extraordinary wave in the ionospheric plasma fits on these circles. As h increases, the concentration N_e increases, which leads to a rapid decrease in the wavelength. At $h = 95$ km, the wavelength according to Figs. 3 and 4 is 1.5 km. Note that in the interval $60 < h < 80$ km, horizontally polarized waves arise, traveling both upward and downward. The latter is represented on the hodograph $H_\theta(h)$ by a circle of radius $|r_{21}| \cong 0.07$.

Fig. 3

Fig. 4

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Note: Figure translations are in progress. See original paper for figures.