MAPPINGS QUASICONFORMAL ON THE AVERAGE

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.54

MATHEMATICS

I. N. PESIN

MAPPINGS QUASICONFORMAL ON THE AVERAGE

(Presented by Academician M. A. Lavrent' ev, 18 XII 1968)

Let $p(z)$, $\theta(z)$ be the characteristics of a quasiconformal mapping $w(z)$; $p(z)$ is defined almost everywhere; $\theta(z)$ is defined almost everywhere on the set $\{z : p(z) > 1\}$. The classical theory of quasiconformal mappings concerns the case $p(z) < M$, in which there is an existence and uniqueness theorem ([1]). It is known that in the case of an unbounded characteristic $p(z)$ the global properties of the class of mappings are destroyed. This also applies to mappings with characteristic $p(z)$ satisfying the condition

$$\iint p^n(z)\, d\sigma_z < \infty. \tag{1}$$

Whatever natural number n may be, such mappings form a non-uniformly continuous class, in whose closure there are discontinuous mappings. Therefore it has been preferable to consider classes of inverse mappings possessing the property of uniform continuity (see, for example, ([2],[3])). Nevertheless, a global theory can be developed for classes of mappings with an unbounded characteristic satisfying a boundedness condition for a certain integral mean, at least of exponential type ([4]); in this case compact and closed classes of mappings are obtained, for which an existence theorem for prescribed characteristics holds. Questions of uniqueness present very substantial difficulties and are connected with problems of univalent mappings of the classes $ACTL_p$.

It is well known that if, in a neighborhood of an isolated point, the characteristic $p(z)$ does not grow too rapidly, for example, so that

$$\int \frac{dr}{rp(r, z_0)} = +\infty, \qquad p(r, z) = \sup_{|z - z_0| = r} p(z), \tag{2}$$

then the mapping is continuous at the point z_0. The condition of uniform divergence of such an integral (in an appropriate sense) is at the same time a condition of uniform continuity of the class of mappings at the point z_0. In order to formulate a similar condition in a domain, it suffices to replace the requirement of divergence of the integral (2) by the requirement of convergence of a certain area integral (there exists an example showing that a condition of the form $\iint \frac{d\sigma_z}{r^2 F(p(z))} = +\infty$ is not sufficient for continuity of the mapping at the point z_0, whatever positive function $F(x)$ may be). It is sufficient to require, for example, convergence of the integral

$$\iint e^{[p(z)]^{1+\alpha}}\, d\sigma_z$$

for some positive α. Such a requirement is probably not too excessive, if one takes into account that the conditions (1) are insufficient.

Let us give the exact definition of the class of mappings. The class $H(M, \alpha)$ of homeomorphic mappings of the domain D_z satisfies the following requirements: 1) for any $p < 2$ they are $ACTL_p$-mappings in D_z, i.e., mappings absolutely continuous in the sense of Tonelli, and par-

partial derivatives belonging to L_p; 2) the inverse mappings in the image of the domain D_z belong to the class $ACTL_2$. By Morrey' s theorem (^{5}) on absolute continuity in the Banach sense of inverse mappings and Men' shov' s theorem (^{6}) on the differentiability of single-sheeted mappings with finite derivatives, the direct mappings are nondegenerately differentiable almost everywhere, and hence the characteristics $p(z), \theta(z)$ arise; 3) $p(z)$ satisfies the condition

$$\iint_{D_z} e^{p^{1+\alpha}}\, d\sigma_z < M. \tag{3}$$

Let us note that the class $H(M, \alpha)$ is not changed under supplementary conformal mappings of the domain $D_\eta = f(D_z)$, since $p(z)$ is then not changed, while the norms of the partial derivatives of the direct and inverse mappings in D_z and in $f(D_z)$, respectively, are estimated in terms of the norms $p(z)$ and the areas of the domain.

Lemma. Let D_z be the disk $|z| < 1,\ 0 < r < 1$. Then

$$\iint_{|z|>r} p(z)\, d\sigma_{\ln z} \leq C_1(M) + C_2(\alpha) \left[\ln \frac{1}{r}\right]^{(2+\alpha)/(1+\alpha)}; \tag{4}$$

C_1, C_2 are constants depending on M and α, respectively.

We now consider mappings of the class $H(M)$ of the annulus $r < |z| < 1$ onto the annulus $\rho < |w| < 1$. The well-known inequality holds

$$\ln \frac{1}{\rho} \geq \frac{1}{\iint_{r<|z|<1} p(z)\, d\sigma_{\ln z}} \left(\ln \frac{1}{r}\right)^2, \tag{5}$$

and from (4) and (5), for sufficiently small r, we obtain

$$\ln \frac{1}{\rho} \geq C_3(M, \alpha) \left(\ln \frac{1}{r}\right)^{\alpha/(1+\alpha)}. \tag{6}$$

It follows from (6) that mappings of the class $H(M)$ of the disk $|z| < 1$ onto the disk $|w| < 1$, with the normalization $w(0) = 0$, are equicontinuous at $z = 0$. The equicontinuity of these mappings in the disk $|z| < 1$ is established with the aid of known methods, namely: first in the disk $|z| < \tau < 1$, using auxiliary linear transformations in the planes z and w, and then in the whole disk by extending the mapping across the circle $|z| = 1$ according to the principle of symmetry and by a subsequent conformal mapping. Here it is taken into account that the constant M changes in a bounded way and that the inverse mappings form an equicontinuous class.

Let a sequence $w_i(z)$ of mappings of the class $H(M, \alpha)$, with characteristics $p_i(z)$, of the disk $|z| < 1$ onto the disk $|w| < 1$, converge uniformly to the mapping $w(z)$. From the closure properties of the classes $ACTL_p$ it follows that $w(z)$ satisfies requirements 1), 2) in the definition of $H(M, \alpha)$. Fulfillment of condition 3) follows from the relation

$$\lim_{i\to\infty} \iint p_i^n(z)\, d\sigma_z \geq \iint p^n\, d\sigma_z,$$

where $p(z)$ is the characteristic of the limiting mapping $w(z)$ ([2]). Thus we have proved

Theorem 1. The class of mappings $H(M, \alpha)$ of the disk $|z| < 1$ onto the disk $|w| < 1$, with the normalization $w(0) = 0$, is equicontinuous and closed with respect to uniform convergence.

The fact that the class $H(M, \alpha)$ is essentially broader than the class of quasi-conformal mappings with bounded characteristic is shown by the following

Theorem 2. Let $p(z), \theta(z)$ be a distribution of characteristics in the disk $|z| < 1$, $\alpha > 0$, and

$$\iint e^{p^{1+\alpha}}\, d\sigma_z < M.$$

Then there exists a mapping

$w(z)$ of the disk $|z| < 1$ onto the disk $|w| < 1$ of the class $H(M, \alpha)$, with characteristics $p(z), \theta(z)$ almost everywhere.

Proof. Consider the truncated functions $p_N(z)$ and the sequence of quasiconformal mappings $w = f_N(z)$ of the disk $|z| < 1$ onto the disk $|w| < 1$, $f_N(0) = 0$, with bounded characteristics $p_N(z), \theta(z)$. By Theorem 1 one may assume that the sequence $f_N(z)$ converges uniformly to a mapping $f(z)$ of the class $H(M, \alpha)$. $f(z)$ is the desired mapping: namely, it is almost everywhere nondegenerately differentiable, and its characteristics are the required functions $p(z)$ and $\theta(z)$ at all Lebesgue points of the functions $p(z)$ and $\theta(z)$, and at almost all points where $p(z) = 1$. The corresponding proof differs from the proof of the existence theorem in [(1)] only in inessential details.

As for the uniqueness theorem, one can prove the following: in order that a mapping with prescribed characteristics $p(z), \theta(z)$ be unique in the class $H(M, \alpha)$, it is necessary that it be absolutely continuous in the sense of Banach. In this connection we note the unsolved problem of absolute continuity in the sense of Banach of univalent mappings of the class $ACTL_p$, $p < 2$.

Lviv State University
named after Iv. Franko

Received
21 III 1968

REFERENCES CITED

[1] P. P. Belinskii, I. N. Pesin, DAN, **102**, No. 5, 865 (1955). [2] L. Ahlfors, J. Anal. Math., **3**, 1 (1953-1954). [3] S. L. Krushkal, DAN, **157**, No. 3, 517 (1961). [4] I. N. Pesin, Abstracts of the XL Scientific Conf. of Lviv State Univ., Lviv, 1960, p. 11. [5] C. B. Morrey, Trans. Am. Math. Soc., **43**, 126 (1938). [6] D. Menchoff, Math. Ann., **105**, 75 (1931).

Note: Figure translations are in progress. See original paper for figures.