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A. I. OKSAK

ON THE NORMAL FORM OF OPERATORS IN FOCK SPACE

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In the paper certain classes of unbounded operators in the space of second quantization are indicated which can be reduced to normal form. The possibility of such a reduction is closely connected with the domains of definition of the operators and of their adjoints. We shall confine ourselves to the study of operators in Bose space; the generalization to the fermion case presents no difficulties.

1. **Definition 1.** Let C be a linear operator with domain of definition $D(C)$ in a Hilbert space H and range $R(C)$ in a Hilbert space $\mathcal{H}$; let H' and $\mathcal{H}'$ be (closed) subspaces in H and $\mathcal{H}$, respectively; and let P and $\mathcal{P}$ be projection operators: $H' = PH$, $\mathcal{H}' = \mathcal{P}\mathcal{H}$. By the part of the operator C belonging to the subspaces H' and $\mathcal{H}'$ we shall mean an operator X with $D(X) \subset H'$, $R(X) \subset \mathcal{H}'$, such that

$$D(X) = D(C) \cap H',$$

$$Xf = \mathcal{P}Cf \equiv \mathcal{P}CPf \quad \text{for } \forall f \in D(X).$$

Let us note that a part X of a closed, densely defined operator C, generally speaking, need not admit a closure. However, if the part Y of the operator C^*, belonging to the subspaces $\mathcal{H}'$ and H', has a dense domain of definition $D(Y) \subset \mathcal{H}'$, then the part X of the operator C, belonging to the subspaces H' and $\mathcal{H}'$, admits a closure, and $\overline{X} \subset Y^*$. If $D(X)$ is dense in H', then Y admits a closure and $\overline{Y} \subset X^*$.

2. Let D_i be an arbitrary linear manifold in the Hilbert space H_i ($i = 1, \dots, n$); $L(D_1, \dots, D_n)$ will henceforth denote the linear span of vectors of the form $f_1 \otimes \cdots \otimes f_n$ with arbitrary $f_i \in D_i$, in the tensor product $H_1 \otimes \cdots \otimes H_n$ (see ([1])).

Definition 2 ([2]). The tensor product $A_1 \otimes \cdots \otimes A_n$ of closed (or closable), densely defined (i.e., with dense domains of definition in the corresponding spaces) operators $A_i : D(A_i) \subset H_i,\ R(A_i) \subset \mathcal{H}_i,\ i = 1, \dots, n$, is the closure of the operator A, defined on $D(A) = L(D(A_1), \dots, D(A_n)) \subset H_1 \otimes \cdots \otimes H_n$ with values in $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ by the formula:

$$A(f_1 \otimes \cdots \otimes f_n) = (A_1 f_1) \otimes \cdots \otimes (A_{nf} n).$$

Lemma 1. *Let A and B be closed operators with dense domains of definition in one and the same Hilbert space $D(A) \subset H$, $D(B) \subset H$, and with ranges $R(A) \subset \mathcal{H}_1$, $R(B) \subset \mathcal{H}_2$, where $D(A) \supset D(B)$; let S be a closed densely defined operator, $D(S) \subset H'$, $R(S) \subset \mathcal{H}'$. Then $D(S \otimes A) \supset D(S \otimes T)$, where $T = \sqrt{1 + B^* B}$ is the positive square root of the self-adjoint operator $1 + B^* B$. If, in addition, one of the requirements is satisfied: (a) S is bounded, (b) B has a bounded inverse, then*

$$D(S \otimes A) \supset D(S \otimes B). \tag{1}$$

From Lemma 1 there follows the well-known fact: the tensor product of bounded operators S and A is a bounded operator; it is enough in (1) to put $B = 0$: $D(S \otimes A) \supset D(S \otimes 0) = H' \otimes H$; see ([2]), Theorem 2.

3. Denote by $H^{\otimes n}$ the n-th tensor power of the space H (i.e., the tensor product of n copies of the space H); $H^{\vee n}$ is the n-th symmetric power of H—this is the subspace in $H^{\otimes n}$ that is the range of the symmetrization operator Q_n in $H^{\otimes n}$ (see (3), Appendix A). Below, by symbols of the type $A^{(m,n)}$ we shall denote closed (or closable) operators with dense domains of definition $D(A^{(m,n)}) \subset H^{\vee n}$ and ranges $R(A^{(m,n)}) \subset H^{\vee m}$. As follows from Sec. 2, the expression $A^{(m,n)} \otimes B^{(m',n')}$ defines a closed operator with dense domain of definition $D(A^{(m,n)} \otimes B^{(m',n')}) \subset H^{\vee n} \otimes H^{\vee n'}$ and with $R(A^{(m,n)} \otimes B^{(m',n')}) \subset H^{\vee m} \otimes H^{\vee m'}$. It is obvious that $H^{\vee(n+n')}$ and $H^{\vee(m+m')}$ are (closed) subspaces, respectively, in $H^{\vee n} \otimes H^{\vee n'}$ and $H^{\vee m} \otimes H^{\vee m'}$; therefore we can give the following definition (see also Sec. 1):

Definition 3. The symmetric (tensor) product $A^{(m,n)} \vee B^{(m',n')}$ of the operators $A^{(m,n)}$ and $B^{(m',n')}$ is the part of the operator $A^{(m,n)} \otimes B^{(m',n')}$ belonging to the subspaces $H^{\vee(n+n')}$ and $H^{\vee(m+m')}$.

From Lemma 1 it follows immediately

Theorem 1. *Let $A^{(m,n)}$, $B^{(m',n)}$, $S^{(k,l)}$ be closed operators with dense domains of definition respectively in $H^{\vee n}$, $H^{\vee n}$, $H^{\vee l}$ and with values respectively in $H^{\vee m}$, $H^{\vee m'}$, $H^{\vee k}$, and suppose that $D(A^{(m,n)}) \supset D(B^{(m',n)})$. Assume additionally that $S^{(k,l)}$ is bounded. Then*

$$D(S^{(k,l)} \vee A^{(m,n)}) \supset D(S^{(k,l)} \vee B^{(m',n)}).$$

It is obvious that the theorem remains valid if, instead of the closed operator $A^{(m,n)}$, one takes the closable operator $A^{(m,n)}$.

Lemma 2. *If the operators* $X = A^{(m,n)} \vee B^{(m',n')}$ *and* $Y = A^{(m,n)*} \vee B^{(m',n')*}$ *have dense domains of definition respectively in* $H^{\vee(n+n')}$ *and* $H^{\vee(m+m')}$*, then they admit closures, with* $\overline{X} \subset Y^*$*,* $\overline{Y} \subset X^*$*.*

4. We turn to the consideration of operators in Fock space. H is the Hilbert space of one-particle states, $H^{(n)} = H^{\vee n}$ is the n-th symmetric power of H;

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} H^{(n)}$$

is the Fock space of Bose particles. Let M be a self-adjoint operator in $\mathcal{H}$ with domain of definition $D(M)$, commuting with the projectors onto the subspaces $H^{(n)} \subset \mathcal{H}$; $M^{(n,n)}$ is the part of the operator M lying in $H^{(n)}$ with domain of definition

$$G^{(n)} = D(M^{(n,n)}). \tag{2}$$

We impose on the operator M the following essential condition: for any $s, k = 0, 1, ...$

$$D(I^{(s,s)} \vee M^{(k,k)}) \supset D(M^{(s+k,s+k)}) = G^{(s+k)}, \tag{3}$$

where $I^{(s,s)} \vee M^{(k,k)}$ is the symmetric product of the identity operator $I^{(s,s)}$ in the space $H^{(s)}$ with the operator $M^{(k,k)}$ (see Definition 3).

Starting from an operator M satisfying condition (3), we construct a class of operators in $\mathcal{H}$ that expand into a series in creation and annihilation operators. To this end we introduce the space $\Theta = \Theta_M$ of coefficient functions—this is the set of matrices $K = [K^{(m,n)}]_{m,n=0,1,...}$, whose elements are operators satisfying the condition

$$D(K^{(m,n)}) \supset G^{(n)}, \qquad D(K^{(m,n)*}) \supset G^{(m)} \tag{4}$$

(from (4) it follows that $K^{(m,n)}$ admit closures).

By Theorem 1, from (4), (3) it follows that

$$D(I^{(s,s)} \vee K^{(m,n)}) \supset G^{(s+n)}, \qquad D(I^{(s,s)} \vee K^{(m,n)*}) \supset G^{(s+m)},$$

whence, with the aid of Lemma 2, we establish that $I^{(s,s)} \vee K^{(m,n)}$ admits a closure and

$$D(I^{(s,s)} \vee K^{(m,n)}) \supset G^{(s+n)}, \qquad D((I^{(s,s)} \vee K^{(m,n)})^*) \supset G^{(s+m)}. \tag{5}$$

We shall denote by $\mathcal{D}$ the set of finite (i.e., with a finite number of particles) vectors from $D(M)$:

$$\mathcal{D} = \{\Phi : \ \Phi \in D(M), \ E_n\Phi = 0 \text{ for } n > N(\Phi)\}. \tag{6}$$

To each operator in Fock space such that

$$D(C) \supset \mathcal{D}, \qquad D(C^*) \supset \mathcal{D}, \tag{7}$$

there corresponds a matrix $K \in \Theta$ with elements:

$K^{(m,n)}$ is the part of the operator $(m!n!)^{-1/2} \cdot C$ belonging to the subspaces $H^{(n)}$ and $H^{(m)}$.

(8)

Conversely, in general this is false (a matrix $K \in \Theta$ may correspond to no operator in $\mathcal{H}$).

We shall call a matrix $K \in \Theta$ **Jacobian** if, for some r, $K^{(m,n)} = 0$ for $|m-n| > r$, and **finite** if only a finite number of elements $K^{(m,n)}$ are nonzero.

Lemma 3. To every Jacobian (in particular, finite) matrix $K \in \Theta$ there corresponds (admitting closure) an operator C satisfying (7), (8).

The inclusions (5) make it possible, for each complex number λ, to define a transformation of Θ into itself

$$K \to I_{(\lambda)} * K, \tag{9}$$

where the right-hand side of (9) is a matrix with elements—operators defined on $G^{(n)}$ by the relation:

$$(I_{(\lambda)} * K)^{(m,n)} = \sum_{s=0}^{\min(m,n)} \frac{\lambda^s}{s!} I^{(s,s)} \vee K^{(m-s,n-s)}. \tag{10}$$

(For $\lambda = 0$, $I_{(0)} * K$ is defined to be equal to K.)

The transformations (9) have inverses, as follows from the identity

$$I_{(\lambda)} * [I_{(\mu)} * K] = I_{(\lambda+\mu)} * K. \tag{11}$$

5. For any Jacobian matrix $A \in \Theta$, $I_{(1)} * A$ is a Jacobian matrix from Θ. By Lemma 3 it is associated with an operator B in $\mathcal{H}$, satisfying (7) and the condition:

the part $B^{(m,n)}$ of the operator $(m!n!)^{-1/2} \cdot B$, belonging to the subspaces $H^{(n)}$ and $H^{(m)}$, is equal to

$$(I_{(1)} * A)^{(m,n)} = \sum_{s=0}^{\min(m,n)} \frac{1}{s!} I^{(s,s)} \vee A^{(m-s,n-s)}. \tag{12}$$

Definition 4. The expansion

$$A\{a^+, a^-\} = \sum_{m,n} (a^+)^m \cdot A^{(m,n)} \cdot (a^-)^n \tag{13}$$

for a Jacobian matrix $A \in \Theta$ will be called the operator B corresponding to the Jacobian matrix $I_{(1)} * A$ and, therefore, defined by the equality (12). (We note that essentially in this form the concept of normal form was used by H. Araki for the study of bounded operators ([4], §6).)

We generalize the concept of the expansion (13) to the case of an arbitrary (non-Jacobian) matrix $A \in \Theta$. In this case, by the infinite sum on the right-hand side of (13) one naturally understands the limit

$$B = \lim_{r\to\infty} \sum_{|m-n|\le r} (a^+)^m \cdot A^{(m,n)} \cdot (a^-)^n, \tag{14}$$

if it exists on the set of finite vectors $\mathcal{D} \subset D(M)$.

With the aid of identity (11) one establishes

Theorem 2. Let C be an operator in the Fock space $\mathcal{H}$ satisfying condition (7); let $K = \|K^{(m,n)}\| \in \Theta$ be the corresponding matrix (8). Then, for the matrix $A = I_{(-1)} * K$, expression (14) defines an operator B which is the restriction of the operator C to the domain $\mathcal{D}$:

$$B = C|_D. \tag{15}$$

This theorem on reducing operators to normal form for a bounded operator C was established by F. A. Berezin [5,6] (see also [4], § 6). The question arises whether the expansion (13) for an infinite matrix A can be understood as the limit of repeated sums:

$$B = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} (a^+)^m \cdot A^{(m,n)} \cdot (a^-)^n \right). \tag{16}$$

It turns out that, even for a bounded operator C, the repeated series (16), generally speaking, converges neither in the strong nor in the weak sense (the corresponding example is given in the supplement to [7]); the expansion (16) for an infinite matrix A can be understood in a certain **superweak** sense, see [4,6]).

As an application we consider the out- and in-fields (corresponding to particles of positive mass) of quantum field theory with the condition of asymptotic completeness [8]. They are defined on the common domain $D_i(M)$—the domain of definition of the energy operator $M = P_0$, which satisfies the basic condition

(3). This makes it possible to conclude that the out-fields expand in a series in terms of in-fields.

Remark. Let the matrix $A \in \Theta$ have the only nonzero element $K^{(1,0)}$ (respectively $K^{(0,1)}$); then to it there corresponds a vector $f \in H$ such that the operator $K^{(1,0)}$, taking the vacuum subspace $H^{(0)}$ into the one-particle subspace $H^{(1)}$, has the form $H^{(0)} \ni c \to K^{(1,0)}c = cf \in H^{(1)}$ (respectively, $K^{(0,1)}$ has the form $H^{(1)} \ni F \to K^{(0,1)}F = (f, F) \in H^{(0)}$). For such a matrix A, expression (13) defines the creation operator $a^+(f)$ (respectively, the annihilation operator $a^-[f]$)—this is an operator-valued linear (respectively antilinear) functional on $f \in H$, where $a^-[f] \subset (a^+(f))^*$, $[a^-[f], a^+(g)] \subset (f, g)$. If $\mathcal{J}$ is a certain involuting operator in H, then instead of $a^-[f]$ one introduces [6] the operator-valued linear functional $a^-(f) = a^-[\mathcal{J}f]$.

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Moscow Institute of Physics and Technology

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Note: Figure translations are in progress. See original paper for figures.