OPTIMALITY CONDITIONS FOR CERTAIN CLASSES OF SYSTEMS NOT RESOLVED WITH RESPECT TO THE DERIVATIVE

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Abstract

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OPTIMALITY CONDITIONS FOR CERTAIN CLASSES OF SYSTEMS NOT RESOLVED WITH RESPECT TO THE DERIVATIVE

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The paper formulates necessary optimality conditions for controlled systems with parameters whose behavior is described by integro-differential equations not resolved with respect to the derivative, as well as by equations with delay of neutral type; classes of systems are indicated for which these conditions are sufficient for optimality. Examples are given showing that, for such problems, the maximum principle, generally speaking, does not apply.

Let E_k be a Euclidean space of dimension k. Let certain sets $U \subset E_r$ and $W \subset E_s$ be given, and let D be the set of vector functions $u(t) = (u^1(t), \dots, u^r(t))$, $t_0 \leqslant t \leqslant T$, with piecewise-continuous and piecewise-differentiable coordinates $u^i(t)$, where $u(t) \in U$ for $t_0 \leqslant t \leqslant T$, and at points of discontinuity $u^i(t)$ are continuous from the left, while at the point t_0 they are continuous from the right; the instants t_0, T are given. On the set of pairs $(u(t), w) \in D \times W$ consider the functional:

$$J(u,w) = \int_{t_0}^{T} f^0(x(t), y_{10}(t), \dot{y}_{20}(t), t, u(t), w)\, dt + \\ + \int_{t_0}^{T} dt \int_{t_0}^{t} g^0(t, x(\eta), y_{30}(\eta), \dot{y}_{40}(\eta), \eta, u(\eta), w)\, d\eta + \Phi(x(T)), \tag{1}$$

where the vector function $x(t) \equiv x(t,u,w) = (x^1(t), \dots, x^n(t))$, $t_0 \leqslant t \leqslant T$, is defined as the solution of the problem:

$$\dot{x}^i(t) = f^i(x(t), y_{1i}(t), \dot{y}_{2i}(t), t, u(t), w) +$$

$$+ \int_{t_0}^{t} g^i(t, x(\eta), y_{3i}(\eta), \dot{y}_{4i}(\eta), \eta, u(\eta), w)\, d\eta, \qquad t_0 \leqslant t \leqslant T, \tag{2}$$

$$x^i(t) = \varphi_i(t), \qquad t_0 - \Delta \leqslant t \leqslant t_0 \quad (i = 1, 2, \dots, n); \tag{3}$$

here

$$y_{\nu i}(t) \equiv (x^1(t - \tau_{\nu i1}(t)), \dots, x^n(t - \tau_{\nu in}(t))),$$
$$\dot{y}_{\nu i}(t) \equiv (\dot{x}^1(t - \tau_{\nu i1}(t)), \dots, \dot{x}^n(t - \tau_{\nu in}(t))),$$

where $\dot{x}^j(t - \tau_{\nu ij}(t))$ is understood as the value of the derivative $\dot{x}^j(\xi)$ at $\xi = t - \tau_{\nu ij}(t)$. Everywhere below it is assumed that $\varphi_j(t)$ are continuous and twice piecewise-differentiable functions on the interval $t_0 - \Delta \leqslant t \leqslant t_0$,

$$\Delta = \max_{\nu,i,j} \max_{t_0 \leqslant t \leqslant T} \tau_{\nu ij}(t);$$

$\tau_{\nu ij}(t)$ are given nonnegative continuous functions with piecewise-continuous first derivatives, $\dot{\tau}_{\nu ij}(t \pm 0) \leqslant 1 - \alpha_{\nu ij}$, $\alpha_{\nu ij} = \text{const} > 0$, while $\tau_{2ij}(t)$, $\tau_{4ij}(t)$ have second piecewise-continuous derivatives; $\Phi(x)$, $f^i(x, y, z, t, u, w)$, $g^i(t, x, y, z, \eta, u, w)$ are given functions, continuous in the totality of their arguments together with their first partial derivatives with respect to the variables x, y, z, u, w, t, and, moreover, the partial derivatives $f^i_{z^j}(x, y, z, t, u, w)$, $g^i_{z^j}(\eta, x, y, z, t, u, w)$ are continuously differentiable with respect to x, y, z, t, u; here $(x, y, z, u, w) \in E_n \times E_n \times E_n \times U \times W$, $t_0 \leqslant \eta, t \leqslant T$, $i = 0, 1, \dots, n$, $j = 1, 2, \dots, n$; $\nu = 1, 2, 3, 4$.

By a solution of problem (2), (3) corresponding to the pair $(u(t), w) \in D \times W$, we shall understand a vector function $x(t) \equiv x(t, u, w)$ with continuous and piecewise-differentiable coordinates, satisfying

system (2) at all points of continuity of the derivatives entering the system and the initial conditions (3). The existence and uniqueness of the solution of problem (2), (3) for fixed $(u(t), w)$ can be proved by the methods of papers ($^{1-3}$), if, for example, it is additionally assumed that the functions f^i, g^i $(i = 1, 2, \dots, n)$ satisfy a Lipschitz condition with respect to the variables (x, y) and, moreover, either a) $\tau_{2ij}(t) > 0$, or

b) $\tau_{2ij}(t) \equiv 0$, $\max_{1 \le j \le n} \sum_{i=1}^n L^2_{ji} < n^{-2}$, $L_{ji} = \sup |f^j_{z^i}|$, or some other conditions.

We shall consider the following optimal control problem: among all $(u(t), w) \in D \times W$ find a pair $(u^*(t), w^*)$ such that

$$J(u^*, w^*) = \inf_{D \times W} J(u, w). \tag{4}$$

A pair $(u^*(t), w^*) \in D \times W$ satisfying condition (4) will be called an optimal solution of problem (1)–(4).

Define: $r_{\nu ji}(\xi) \equiv \gamma_{\nu ji}(\xi)$ for $t_0 - \tau_{\nu ji}(t_0) \le \xi \le T - \tau_{\nu ji}(T)$; $r_{\nu ji}(\xi) \equiv T$ for $T - \tau_{\nu ji}(T) \le \xi \le T$, where $\gamma_{\nu ji}(\xi)$ is the inverse function for $\xi_{\nu ji}(t) \equiv t - \tau_{\nu ji}(t)$.

Let $\psi(t) \equiv \psi(t, u, w) = (\psi_1(t), \dots, \psi_n(t))$, $t_0 \le t \le T$, be a piecewise-continuous and piecewise-differentiable vector function which may have discontinuities at those points $t = \theta$, $t_0 \le \theta < T$, at which $\dot{y}_{\nu j}(r_{\nu ji}(t))$, $u(r_{\nu ji}(t))$, $\dot{r}_{\nu ji}(t)$ $(\nu = 2, 4)$, have discontinuities, and at points of the form $\xi_{2i_1j_1}(\xi_{2i_2j_2}(\dots \xi_{2i_{kk}j}(\theta)\dots))$ for arbitrary $k = 1, 2, \dots$, $1 \le i_k, j_k \le n$; at the points of continuity of $\psi(t)$, $\psi(r_{2ji}(t))$, the equations

$$\begin{aligned}
\dot{\psi}_i(t) = -\sum_{j=0}^{n} \Bigg[& \psi_j(t) f^j_{x^i}(x(t), y_{1j}(t), y_{2j}(t), t, u(t), w) \\
& + \int_t^T \psi_j(\eta) g^j_{x^i}(\eta, x(t), y_{3j}(t), y_{4j}(t), t, u(t), w)\, d\eta \\
& + \dot{r}_{1ji}(t) \psi_j(\xi) f^j_{y^i}(x(\xi), y_{1j}(\xi), y_{2j}(\xi), \xi, u(\xi), w)\big|_{\xi = r_{1ji}(t)} \\
& + \dot{r}_{3ji}(t) \int_{r_{3ji}(t)}^T \psi_j(\eta) g^j_{y^i}(\eta, x(\xi), y_{3j}(\xi), y_{4j}(\xi), \xi, u(\xi), w)\big|_{\xi = r_{3ji}(t)}\, d\eta \Bigg] \\
& + \frac{d}{dt} \Bigg\{ \sum_{j=0}^{n} \Bigg[\dot{r}_{2ji}(t) \psi_j(\xi) f^j_{z^i}(x(\xi), y_{1j}(\xi), y_{2j}(\xi), \xi, u(\xi), w)\big|_{\xi = r_{2ji}(t)} \\
& + \dot{r}_{4ji}(t) \int_{r_{4ji}(t)}^T \psi_j(\eta) g^j_{z^i}(\eta, x(\xi), y_{3j}(\xi), y_{4j}(\xi), \xi, u(\xi), w)\big|_{\xi = r_{4ji}(t)}\, d\eta \Bigg] \Bigg\};
\end{aligned} \tag{5}$$

at the points of discontinuity $t = \theta$ the function $\psi(t)$ has a jump

$$[\psi_i(t)]_\theta = \Bigg[\sum_{j=0}^{n} \Bigg[\dot{r}_{2ji}(t) \psi_j(\xi) f^j_{z^i}(x(\xi), y_{1j}(\xi), y_{2j}(\xi), \xi, u(\xi), w)\big|_{\xi = r_{2ji}(t)} + \dot{r}_{4ji}(t) \int_{r_{4ji}(t)}^T \psi_j(\eta) g^j_{z^i}(\eta, x(\xi), y_{3j}(\xi), y_{4} \tag{6}$$

the initial conditions are satisfied:

$$\psi_i(T) = -\Phi_{x^i}(x(T)) + \sum_{j=0}^{n} \psi_j(T) f^j_{z^i}(x(T), y_{1j}(T), y_{2j}(T), T, u(T), w) \dot{r}_{2ji}(T) \tag{7}$$

$(i = 1, 2, \dots, n)$; here $\psi_0(t) \equiv -1$, $x(t) = x(t, u, w)$, $[z(t)]_0 = z(\theta + 0) - z(\theta - 0)$. Introduce the function

$$H\{x(t), \dot{x}(t), \psi(t), t, u, w\} = \sum_{j=0}^{n} \Bigg[\psi_j(t) f^j(x(t), y_{1j}(t), y_{2j}(t), t, u, w) + \int_t^T \psi_j(\eta) g^j(\eta, x(t), y_{3j}(t), y_{4j}(t), t, u, w)\, d$$

Theorem 1. Let $(u^*(t), w^*) \in D \times W$ be an optimal solution of problem (1) –(4), and let $x^*(t) = x(t, u^*, w^*)$, $\psi^*(t) = \psi(t, u^*, w^*)$ be the corresponding solutions of problems (2)–(3) and (5)–(7). Then it is necessary that:

1) The relation

$$\overline{\lim}_{k\to\infty} \sum_{i=1}^{r} \frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial u^i} \frac{\Delta u^i(\varepsilon_k, t)}{\varepsilon_k} \le 0, \qquad t_0 \le t \le T, \tag{8}$$

hold for any sequence of functions $\Delta u(\varepsilon_k, \xi)$ $(k = 1, 2, ...)$ possessing the properties: a) $\Delta u(\varepsilon_k, \xi)$, for each $k = 1, 2, ...$, is defined, piecewise continuous, and piecewise differentiable with respect to ξ on some interval $t' \le \xi \le t'' = t$ (on the interval $t' = t \le \xi \le t''$ when $t = t_0$), $t_0 \le t' < t'' \le T$, and $\Delta u(\varepsilon_k, \xi)$ is left-continuous for every ξ, $t' < \xi \le t''$, and right-continuous at $\xi = t'$, continuously at $\xi = t$ uniformly in k; b) $u^*(\xi) + \Delta u(\varepsilon_k, \xi) \in U$ for $t' \le \xi \le t''$ and $\sup_{t' \le \xi \le t''} |\Delta u(\varepsilon_k, \xi)| \le C\varepsilon_k$, $C = \text{const} > 0$ $(k = 1, 2, ...)$; c) $\varepsilon_k > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$; moreover, if $u^*(t)$ is an interior point of the set U, then

$$\frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial u^i} = 0 \qquad (i = 1, 2, \dots, r). \tag{9}$$

In particular, if U is a convex set or is star-shaped with respect to the points $u^*(\xi)$, then relations (8), (9) may be written in the form

$$\max_{u\in U} \left\{ \sum_{i=1}^{r} \frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial u^i} (u^i - u^{i*}(t)) \right\} = 0, \qquad t_0 \le t \le T. \tag{10}$$

2) The relation

$$\overline{\lim}_{k\to\infty} \sum_{i=1}^{s} \frac{\Delta w^i(\varepsilon_k)}{\varepsilon_k} \int_{t_0}^{T} \frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial w^i}\, dt \le 0 \tag{11}$$

hold for any sequence $\Delta w(\varepsilon_k)$ $(k = 1, 2, \dots)$ possessing the properties: a) $w^* + \Delta w(\varepsilon_k) \in W$ and $|\Delta w(\varepsilon_k)| \le C\varepsilon_k$, $C = \text{const} > 0$ $(k = 1, 2, \dots)$; b) $\varepsilon_k > 0$ $(k = 1, 2, \dots)$, $\lim_{k\to\infty} \varepsilon_k = 0$; moreover, if w^* is an interior point of the set W, then

$$\int_{t_0}^{T} \frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial w^i}\, dt = 0 \qquad (i = 1, 2, \dots, s). \tag{12}$$

In particular, if W is a convex set or is star-shaped with respect to the point w^*, then relations (11), (12) may be written in the form

$$\max_{w \in W} \left\{ \sum_{i=1}^{s} (w^i - w^{i*}) \int_{t_0}^{T} \frac{\partial H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u^*(t), w^*)}{\partial w^i} \, dt \right\} = 0. \qquad (13)$$

Theorem 2. Let U, W be convex sets, the functions Φ, f^0, g^0 be jointly convex in (x, y, z, u, w), and the functions f^i, g^i $(i = 1, 2, \dots, n)$ be linear with respect to x, y, z, u, w. Then for the optimality of $(u^*(t), w^*)$, fulfillment of conditions (10), (13) of Theorem 1 is necessary and sufficient.

For some classes of systems (2) the maximum principle is valid.

Theorem 3. Let

$$f^j \equiv \sum_{i=1}^{n} a_{ji}(t) z^i + f_1^j(x, y, t, u, w),$$

$$g^j \equiv \sum_{i=1}^{n} \bar{a}_{ji}(t, \eta) z^i + g_1^j(t, x, y, \eta, u, w) \qquad (j = 0, 1, \dots, n);$$

$$\sqrt{n} L_1 < 1, \qquad L_1 = \max_{1 \le j \le n} \left(\sum_{i=1}^{n} L_{ji}^2 \alpha_{2ji}^{-2} \right)^{1/2}.$$

Then, for optimality of $(u^*(t), w^*)$, it is necessary that, for every t, $t_0 \le t \le T$, the function $H(x^*(t), \dot{x}^*(t), \psi^*(t), t, u, w)$ of the variable u attain its maximum on U at $u = u^*(t)$ and that assertions (11)–(13) of Theorem 1 hold.

Theorem 4. Let $f^j \equiv b_j(t, u) + f_1^j(x, y, z, t, w)$; $g^j \equiv \bar{b}_j(t, \eta, w) + g_1^j(t, x, y, z, \eta, w)$; Φ, f_1^j, g_1^j $(j = 0, 1, \dots, n)$ be linear with respect to (x, y, z, w); $\sqrt{n} L_1 < 1$. Then, for optimality of $(u^*(t), w^*)$, the assertions of Theorem 3 are necessary and sufficient.

In the general case, for systems of the form (2), as it turns out, the maximum principle does not hold. We give examples.

Example 1. Let $\dot{x}(t) = u(t)$, $\dot{y}(t) = 2\dot{x}^2(t - \tau) - u^3(t)$, $0 \le t \le T = 2\tau$, $\tau = \text{const} > 0$; $x(t) \equiv y(t) \equiv 0$ for $-\tau \le t \le 0$; $J(u) = y(T)$, $U = \{u : |u| \le u_0 < 2\}$. Here $u^*(t) \equiv 0$ for $0 \le t \le \tau$ and $u^*(t) = u_0$ for $\tau \le t \le T$, $\psi_x^*(t) \equiv 0$, $\psi_y^*(t) \equiv -1$, $H \equiv u^3$; on the optimal control, for $0 \le t \le \tau$, the function H attains neither a maximum nor a minimum in U.

Example 2. If in Example 1 we take $U = \{u : 1 \le u \le 2\}$, then $u^*(t) \equiv 2$, $0 \le t \le T$; $\psi_x^*(t) \equiv -8$ for $0 \le t \le \tau$, $\psi_y^*(t) \equiv -1$, $H \equiv u^3 - 8u$ for $0 \le t \le \tau$, and

$H \equiv u^3 - 8$ for $\tau \leq t \leq T$; on the optimal control, for $0 \leq t \leq \tau$, the function H has a local maximum.

Remark 1. For the approximate solution of problem (1)–(4), known methods of gradient type may be applied. The gradient of the functional (1) at the point $(u(t), w)$ in the direct product $L_\infty \times E_s$ is the pair

$$\{-H_u(x(t), \dot{x}(t), \psi(t), t, u(t), w),\ -H_w(x(t), \dot{x}(t), \psi(t), t, u(t), w)\},$$

where $x(t) = x(t, u, w)$, $\psi(t) = \psi(t, u, w)$, $H_u = (H_{u^1}, \dots, H_{u^r})$, $H_w = (H_{w^1}, \dots, H_{w^s})$, and L_∞ is the Banach space of vector-functions $u(t)$ with norm $\sup_{t_0 \leq t \leq T} |u(t)|,\ |u(t)|^2 = \sum_{i=1}^r |u^i(t)|^2$.

Remark 2. If in (1), (2) all $\tau_{\nu ji}(t) \equiv 0$, then we obtain a system without delay, not resolved with respect to the derivatives. The optimality conditions for such systems are expressed by Theorems 1–4 (without any changes in their formulations); in (5)–(7), in this case one should set $\tau_{\nu ji}(t) \equiv 0,\ r_{\nu ji}(t) \equiv t,\ \dot{r}_{\nu ji}(t) \equiv 1,\ f^i_{y^j} \equiv g^i_{y^j} \equiv 0$.

Remark 3. Problems of type (1)–(4) are studied analogously when the functions f^i, g^i depend on an arbitrary finite set of lagging vectors $y_{\nu j}(t),\ \dot{y}_{\nu j}(t)$ $(\nu = 1, 2, \dots, k)$, obtained from the vector $x(t)$ by means of different delay matrices.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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