# ACCOUNTING FOR THE INTERACTION OF THE BOUNDARY LAYER AND THE FREE ATMOSPHERE IN PROGNOSTIC PROBLEMS OF MESOMETEOROLOGY

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**Abstract**

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# ACCOUNTING FOR THE INTERACTION OF THE BOUNDARY LAYER AND THE FREE ATMOSPHERE IN PROGNOSTIC PROBLEMS OF MESOMETEOROLOGY

Atmospheric motions of mesoscales, with characteristic formation intervals of the order of several hours and characteristic horizontal lengths of the order of 1–100 km, should always be considered as occurring against the "background" of large-scale motions with characteristic times of the order of days and lengths of the order of thousands of kilometers. Therefore, in studying mesoprocesses it is convenient to represent certain meteorological elements (for example, pressure and temperature) as the sum of two parts: a quasi-stationary part, describing the "background," and a perturbation from it generated by mesoscale motions (see ([1])). In this note we shall assume that the "background" is representable as a rectilinear motion along the $X_1$ axis with velocity $U_1$.

We start from the system of equations

$$\frac{\partial m}{\partial t}+M\frac{\partial m}{\partial x}+\frac{\partial \varphi}{\partial x}-\nu\frac{\partial^2 m}{\partial z^2}-ln-\Gamma_1\frac{\partial \varphi}{\partial z}-F(m)\equiv A_1; \tag{1}$$

$$\frac{\partial n}{\partial t}+M\frac{\partial n}{\partial x}+\frac{\partial \varphi}{\partial y}-\nu\frac{\partial^2 n}{\partial z^2}+lm-l\rho M-\Gamma_2\frac{\partial \varphi}{\partial z}-F(n)\equiv A_2; \tag{2}$$

$$\frac{\partial \omega}{\partial t}+M\frac{\partial \omega}{\partial x}-\frac{\partial \varphi}{\partial z}+\vartheta-\nu\frac{\partial^2 \omega}{\partial z^2}=-\Gamma_1\frac{\partial \varphi}{\partial x}-\Gamma_2\frac{\partial \varphi}{\partial y}-l(m\Gamma_2-n\Gamma_1)+\Gamma_2\rho M-$$

$$-(\Gamma_1^2+\Gamma_2^2)\frac{\partial \varphi}{\partial z}+\rho\left(u^2\frac{\partial \Gamma_1}{\partial x}+2uv\frac{\partial \Gamma_1}{\partial y}+v^2\frac{\partial \Gamma_2}{\partial y}\right)-F(\omega)\equiv A_3; \tag{3}$$

$$\partial m/\partial x+\partial n/\partial y-\partial \omega/\partial z=0; \tag{4}$$

$$\frac{\partial\vartheta}{\partial t} + M\frac{\partial\vartheta}{\partial x} - D^2\omega - \nu\frac{\partial^2\vartheta}{\partial z^2} = -D^2(m\Gamma_1 + n\Gamma_2) - F(\vartheta) \equiv A_4. \tag{5}$$

Here $x = H^{-1}x_1$; $y = H^{-1}y_1$; $z = H^{-1}(z_1 - z_c)$, where $x_1, y_1$ are dimensional horizontal coordinates; $z_1$ is the dimensional vertical coordinate; $z_1 = z_0(x_1, y_1)$ is the equation of the Earth' s surface; $H$ is a characteristic length; $t = UH^{-1}t_1$; $t_1$ is dimensional time; $U$ is a characteristic velocity; $u = U^{-1}u_1$; $v = U^{-1}v_1$, $\omega = -\rho U^{-1}(w_1 - u_1\Gamma_1 - v_1\Gamma_2)$, where $u_1, v_1, \omega_1$ are dimensional velocities; $\rho = \rho_{01}^{-1}\rho_1$, where $\rho_1$ is dimensional density (approximately the standard density, a function of $z_1$); $\Gamma_1 = \partial z_c/\partial x_1$; $\Gamma_2 = \partial z_0/\partial y_1$; $m = \rho u$; $n = \rho v$; $\varphi = \rho_{01}^{-1}U^{-2}p'$ ($p'$ is the pressure deviation from the "background" value); $\vartheta = \dfrac{g}{T_0}\dfrac{H}{U^2}\rho T'$ ($T'$ is the temperature deviation from the "background" value). The nonlinear terms of the equations are partly included in the function $F$, where

$$F(f) = \partial f(u - M)/\partial x + \partial fv/\partial y - \frac{\partial}{\partial z}f\frac{\omega}{\rho}. \tag{6}$$

Four dimensionless parameters enter:

$$\nu = \frac{\nu_1}{UH}, \qquad D^2 = \frac{gH^2(\gamma_a - \gamma)}{T_0U^2}, \qquad M = \frac{U_1}{U}, \qquad l = \frac{H}{U}l_1, \tag{7}$$

where $\nu_1$ is the coefficient of turbulent thermal conductivity; $\gamma_a, \gamma$ are adiabatic-

···tic and background temperature gradients, respectively; $T_0$ is the mean air temperature; $l_1$ is the Coriolis parameter*.

As boundary conditions we take

$$\text{at } z = 0 \quad m = n = \omega = 0, \quad \vartheta = \theta(x, y, t) \tag{8}$$

($\theta$ is a known function), and the functions tend to zero as $z \to \infty$. At the initial instant all functions are prescribed: $(m)_{t=0} = m^0$, etc.**

We shall assume that for $x = -\infty$, $z_0 = \Gamma_1 = \Gamma_2 = \vartheta = 0$. We shall solve our nonlinear problem by iterations. For the first iteration we take $A_1, \dots, A_4$ to be known functions of the coordinates and time and solve the system of linear equations (1)–(5) with the initial and boundary conditions of the problem. As the known $A_1, \dots, A_4$, for the first iteration one may take the results of linearizing the right-hand sides of our equations with respect to the flow $U_1$.***

For the second iteration we introduce into $A_1, \dots, A_4$ the values of the functions of the coordinates and time $\omega, \vartheta, \varphi, m, n$ obtained in the first iteration, and so on.

To investigate the interaction of atmospheric layers with different character of variation with height (boundary layer, free atmosphere), we use a method analogous to the Fourier method. Consider on the entire semi-infinite axis $0 \le z \le \infty$ an infinite series of points $z_k$ $(k = 0, 1, \ldots, \infty)$ such that $z_0 = 0, \ldots, z_k = k, \ldots$ (in dimensional lengths the interval is equal to $H$). We shall seek the solution only at the nodes of our grid. Let $f(x, y, z_k, t) = f_k(x, y, t)$. First, from (1)–(5), we form differential equations for the Fourier coefficients defined by the relations

$$f_k = \frac{2}{\pi} \int_0^\pi f^\lambda \sin k\lambda \, d\lambda; \qquad f^\lambda = \sum_{k=1}^{\infty} f_k \sin k\lambda. \tag{9}$$

In doing so we take $\partial^2 f/\partial z^2 = (\delta^2 f)_k = f_{k+1} - 2f_k + f_{k-1}$, and then, by (9), $(\delta^2 f)^\lambda = f_0 \sin\lambda - 4\sin^2(\lambda/2) f^\lambda$. Thus, the differential equations for $f^\lambda$ will contain, in particular, the values of the functions $f$ from the boundary condition (8). Excluding in the left-hand sides of the equations for $f^\lambda$ all functions except $\omega^\lambda$, we integrated the corresponding differential equation and then, knowing $\omega^\lambda$, determined $\omega_k$ from (9). We arrived at the formula

$$\omega_k(x, y, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int \sum_{m=0}^{\infty} \frac{K_m(2r)}{m!} \left(\frac{r}{2}\right)^m \sum_{n=0}^{\infty} \binom{m}{n} \left(1 - \frac{l^2}{D^2}\right)^n W_{mnk}(x', y', t) \, dx' \, dy', \tag{10}$$

where

$$W_{mnk}(x, y, t) = \sum_{j=1}^{\infty} \left\{ \alpha_{kj}^{mn}(t) \left[ \Delta\vartheta^0(x - Mt, y) - l\delta \left( \frac{\partial n^0}{\partial x} - \frac{\partial m^0}{\partial y} \right) \right]_j \right.$$

$$\left. - \beta_{kj}^{mn}(t) \nabla^2 \omega_{kj}^0(x - Mt, y) \right\} + \nu \int_0^t \alpha_{k1}^{mn}(t') \, \Delta\theta(x - Mt', y', t - t') \, dt' +$$

* The simplifications made in deriving system (1)–(5) are as follows: a) simplification of the convection theory (equation (5)); b) discarding the term $\partial\rho_1/\partial t$ (equation (4))—filtering acoustic waves; c) simplification of the representation of turbulent viscosity and heat conductivity (constant $\nu_1$; Prandtl number equal to unity, differentiated only with respect to $z_1$); d) the approximation $\rho_1' = \frac{\rho_1}{T_1} T'$ instead of $\rho_1' \simeq \rho_1(p'/p_1 - T'/T_1)$; e) for the background pressure $\varphi_\infty$, it is assumed that $\partial\varphi_\infty/\partial x = 0$, $\partial\varphi_\infty/\partial y = -l\rho M$. The transfer of terms of the type $M\,\partial m/\partial x$ to the left-hand side has been done to accelerate the iteration process.

** We note that $m^0, \dots$ must be related by two diagnostic relations: equation (4) and an equation obtained from (1)–(3) by eliminating time derivatives with the aid of (4).

*** In this case $A_1 = A_2 = 0$, $A_3 = -\rho M^2\, dT_1/dx$, $A_4 = -D^2 \rho M \Gamma_1$.

$$+\sum_{j=1}^{\infty} \int_0^t \left\{ \alpha_{kj}^{mn}(t') \left[ \Delta A_4 - l\delta \left( \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \right]_j - \right.$$

$$\left. -\beta_{kj}^{mn}(t') \left[ \Delta A_3 + \delta \left( \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} \right) \right]_j \right\} dt'. \tag{11}$$

Here $r^2 = (x - x')^2 + (y - y')^2$,

$$\alpha_{kj}^{mn}(a) = e^{-2\nu a} \frac{1}{D} f_{n+1}(Da) \Phi_{kj}^{mn}(2\nu a); \qquad \beta_{kj}^{mn}(a) = e^{-2\nu a} \dot{f}_{n+1}(Da) \Phi_{kj}^{mn}(2\nu a), \tag{12}$$

where

$$f_{n+1}(a) = \frac{1}{2^n n!} \int_0^a a \int_0^a \cdots \int_0^a a \sin a \, (da)^n, \tag{13}$$

$$\Phi_{kj}^{mn}(a) = \sum_{p=0}^{m} \sum_{q=0}^{m-n} \binom{2n}{n+p} \binom{2m-2n}{m-n-q} \frac{(-1)^p}{2^E \left(1 - \frac{p}{n}\right) + E\left(1 - q/(m-n)\right) + m} \times$$

$$\times \left[ I_{p-q-k+j}(a) + I_{p-q+k-j} + I_{p+q-k+j} + I_{p+q+k-j} - I_{p-q-k-j} - \right.$$

$$\left. - I_{p-q+k+j} - I_{p+q-k-j} - I_{p+q+k+j} \right]^* . \tag{14}$$

$$(\delta f)_j = \frac{1}{2}(f_{j+1} - f_{j-1}).$$

Having determined $\omega_k$ by the first iteration, we can then find $\vartheta_k$ from equation (5) in the form:

$$\vartheta_k(x, y, t) = \sum_{j=1}^{\infty} \vartheta_j^0(x - Mt, y) \gamma_{kj}(t) + \nu \int_0^t \gamma_{k1}(t') \theta(x - Mt', y, t - t')\, dt' +$$

$$+\sum_{j=1}^{\infty} \int_0^t \gamma_{kj}(t - t') \left[ A_4(x - M(t - t'), y, t') + D^2 \omega \right]_j dt', \tag{15}$$

where $\gamma_{kj}(a) = e^{-2\nu a}\Phi^{0,0}_{k,j}(2\nu a)$.

Denote $(\partial\varphi/\partial z)_k = S_k$. Then, by (3),

$$S_k = \vartheta_k - A_{3k} + \tilde{\omega}_k, \tag{16}$$

where $\tilde{\omega}_k$ is the result of replacing, in expression (11) for $\omega_k$, the functions $f_{n+1}$ entering into $\alpha, \beta$ by their derivatives with respect to $t$. Knowing $S_k$, we find $\varphi_k$ from the formula

$$\varphi_k = -2\sum_{s=0}^{\infty} S_{2k+s+1}. \tag{17}$$

It remains to find $m_k$ and $n_k$. Using (1) and (2), we obtain

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* In particular, $f_1(a) = \sin a$, $f_2 = \frac{1}{2}(\sin a - a\sin a)$, etc.; $\Phi^{0,0}_{k,j} = I_{k-1} - I_{k+j}$, etc. To compute $I_\nu$ with large indices $\nu$, it is convenient to use formulas expressing $I_\nu$ in terms of $I_0, I_1$:

$$I_{2\nu}(a) = I_0(a)\sum_{k=0}^{\nu-1}\binom{\nu+k-1}{2k}\frac{(\nu+k)!}{(\nu-k)!}\left(\frac{2}{a}\right)^{2k} - I_1(a)\sum_{k=0}^{\nu-1}\binom{\nu+k}{2k-1}\frac{(\nu+k)!}{(\nu-k-1)!}\left(\frac{2}{a}\right)^{2k+1};$$

$$I_{2\nu+1}(a) = -I_0(a)\sum_{k=0}^{\nu-1}\frac{(\nu+k+1)!}{(\nu-k)!}\binom{\nu+k}{2k+1}\left(\frac{2}{a}\right)^{2k+1} + I_1(a)\sum_{k=0}^{\nu-1}\binom{\nu+k}{2k}\frac{(\nu+k)!}{(\nu-k)!}\left(\frac{2}{a}\right)^{2k}.$$

$$\begin{aligned} m_k + in_k = \rho_k M &+ \sum_{j=1}^{\infty}\left[m^0_j(x-Mt,y) - \rho^0_j M + in^0_j\right]\gamma_{kj}e^{-ilt} + \\ &+ \sum_{j=1}^{\infty}\int_0^t \gamma_{kj}(t-t')\left[A_1(x-M(t-t'),y,t') + iA_2 - \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y}\right]_j e^{-ilt'}\,dt' - \\ &- \nu\rho_0 M\int_0^t \gamma_{k1}(t')e^{-ilt'}\,dt'. \end{aligned} \tag{18}$$

An analogous method can be developed in the case when the "background" has a more complex character.

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## CITED LITERATURE

[1] I. A. **Kibel**, *Tr. MMTs*, issue 3, 1964.

*Note: Figure translations are in progress. See original paper for figures.*