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Abstract

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MATHEMATICS

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ON STAR-FINITE COVERINGS AND OPEN-CLOSED SETS

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§ 1. The following results are known.

1. The property of strong paracompactness, generally speaking, is not preserved under perfect (even irreducible) mappings (see ([10])).
2. The property of strong paracompactness is preserved: a) under open perfect mappings (see ([2])); b) under quotient quasi-monotone mappings (see ([3])).
3. A space Y with the first axiom of countability which is a quotient quasi-monotone image of a strongly metrizable space is itself strongly metrizable ([8]).

It turns out that in the results listed above what is important is not that the mapping f is open or quasi-monotone, but that it carries every open-closed set into an open-closed set. For us the following will be fundamental.

Definition. A mapping $f : X \to Y$ will be called a Λ-mapping*, if two conditions are simultaneously fulfilled: a) the image of every open-closed set is an open-closed set; b) for each point $y \in Y$ and every covering ω_y of the set $f^{-1}y$ by open-closed sets in X, there exists a finite subcovering.

Main Theorem 1. *The image of a strongly paracompact (fully paracompact) space X under a Λ-mapping $f : X \to Y$ is again a strongly paracompact (fully paracompact) space.*

Main Theorem 2. *Let $f : X \to Y$ be a quotient Λ-mapping of a strongly metrizable space X onto a space Y with the first axiom of countability**. Then Y is also strongly metrizable.*

§ 2. For the definition of a component of a star-finite covering see ([3–5]).

If η is some system of sets, then by $\tilde{\eta}$ we shall denote the body of this system, i.e. the union of all sets in η. The bodies of different components of a star-countable covering ω do not intersect and are open-closed sets of the space. If $\tau_1, \dots, \tau_s$ are any coverings of a space X, then by $\tau_{12\dots s} = \tau_1 \wedge \dots \wedge \tau_s$ we shall denote the covering consisting of sets that are intersections of one set from each of the coverings $\tau_1, \tau_2, \dots, \tau_s$. All mappings considered by us are continuous, and all spaces are regular. If $f : X \to Y$ is a mapping and $U \subseteq X$, then

$$f^{\#}U = \mathcal{E}(y \in Y,\ f^{-1}y \subseteq U) = Y \setminus f(X \setminus U).$$

§ 3. Proposition 1. *The following properties of a space X are equivalent: 1) X is strongly paracompact. 2) For every open covering ω there exists a disjoint open-closed covering $\Omega = \{\Gamma_\lambda\}$, such that there exist countable closed systems $\xi(\Gamma_\lambda)$, $\Gamma_\lambda = \tilde{\xi}(\Gamma_\lambda)$, such that $\xi = \bigcup_\lambda \xi(\Gamma_\lambda)$ is inscribed in ω.*

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* Every open perfect mapping, as well as every quotient quasi-monotone mapping, is a Λ-mapping. Therefore the results listed above follow automatically from our theorems.

** Theorem 2 is true if one requires only that Y be of point-countable type.

Proof of Main Theorem 1. Let ω be an arbitrary open cover of the space Y. Inscribe in it a cover ω' so that the cover consisting of the closures of its elements is also inscribed in ω. Consider the cover

$$f^{-1}\omega' = \{f^{-1}V,\ V \in \omega'\}$$

and inscribe in it a star-finite cover γ. Consider the components γ_α of the cover, the bodies $\tilde{\gamma}_\alpha$ of these components, and the disjoint open-and-closed cover

$$\delta = \{\tilde{\gamma}_\alpha\}$$

of the space. The cover

$$f\delta = \{f\tilde{\gamma}_\alpha,\ \tilde{\gamma}_\alpha \in \delta\}$$

of the space Y, generally speaking, is not disjoint, nor is it necessarily open-and-closed in Y, because f is a Λ-mapping. For each $y \in Y$ consider the system

$$\delta y = \mathcal{E}(\tilde{\gamma}_\alpha \in \delta,\ \tilde{\gamma}_\alpha \cap f^{-1}y \neq \Lambda).$$

The system δy is finite and consists of open-and-closed sets. Therefore, from the fact that f is a Λ-mapping it follows that the set

$$Wy = f^{\#}\delta y \cap \bigcap_{\tilde{\gamma}_\alpha \in \delta y} \tilde{\gamma}_\alpha$$

is open and closed in Y. Further, it is easily proved that the open-and-closed cover

$$\tau = \{Wy,\ y \in Y\}$$

of the space Y is disjoint and is inscribed in $f\delta$. In addition, the equality

$$f\tilde{\gamma}_\alpha = \bigcup_{U\in\gamma_\alpha} fU = \bigcup_{U\in\gamma_\alpha} [fU]_Y$$

is true. Associate with each element $W \in \tau$ some one element $f\tilde{\gamma}_{\alpha(W)} \in f\delta$ in which W is contained. Denote

$$\xi(W) = \{\, W \cap [fU]_Y,\ U \in \gamma_{\alpha(W)}\,\}, \qquad W \in \tau.$$

The system $\xi(W)$ is countable, consists of closed sets, and $W = \xi(W)$. Put

$$\xi = \bigcup_{\in\tau} \xi(W).$$

The system ξ is inscribed in the cover $f\gamma$, and hence also in ω, and is a closed star-countable cover. Moreover, it has the property that for every W from the disjoint open-and-closed cover τ one necessarily has

$$W = \tilde{\xi}(W),$$

where $\xi(W) \subset \xi$ is a countable subsystem. Thus condition 2 of Proposition 1 is fulfilled, and from it follows the strong paracompactness of the space Y.

p. 4. **Lemma 1 (Stone (9)).** Let $f : X \to Y$ be a quotient mapping and $y_0 \in Y$. Suppose that the first axiom of countability is satisfied at the point y_0. Let $\{\Gamma_n\}$, $n = 1, 2, ...$, be a countable base at the point $y_0 \in Y$; let $\{V_n\}$, $n = 1, 2, ...$, be a countable system of open subsets of X. Suppose the following conditions are satisfied:

$$[\Gamma_{n+1}] \subseteq \Gamma_n, \quad n = 1, 2, \dots; \qquad V_n \subseteq V_{n+1}, \quad n = 1, 2, \dots,$$

$$\dots, \quad f^{-1}y_0 \subseteq \bigcup_{n=1}^{\infty} V_n.$$

Then there is an index n_0 such that

$$\Gamma_{n_0} \subseteq [fV_{n_0}]_Y.$$

From this lemma follows

Lemma 2. Let $f : X \to Y$ be a quotient mapping onto a space with the first axiom of countability. Suppose that for each point $y \in Y$ there is defined a countable finitely additive* system $\Sigma(y)$ of open subsets of X, forming a base in the subspace $f^{-1}y \subseteq X$. Then the system

$$I[f\Sigma(y)] = \{\, I[fU]_Y,\ U \in \Sigma(y)\,\}$$

forms a base at the point y, and the system

$$\bigcup_{y\in Y} I[f\Sigma(y)]$$

is a base in the whole space.

Proposition 2. For the strong metrizability of a space X, it is sufficient and necessary that there exist a σ-star-countable base.

p. 5. **Proof of Main Theorem 2.** Let

$$\sigma = \bigcup_{i=1}^{\infty} \omega_i$$

be a base of the strongly metrizable space X, decomposing into a countable number of star-finite covers ω_i. Consider the components $\omega_{i\alpha}$ of the covers ω_i, the bodies $\tilde{\omega}_{i\alpha}$ of these components; the disjoint open-and-closed covers

$$\delta_i = \{\tilde{\omega}_{i\alpha}\}$$

of the space X, and the open-and-closed (because f is a Λ-mapping) covers

$$f\delta_i = \{f\tilde{\omega}_{i\alpha},\ \tilde{\omega}_{i\alpha} \in \delta_i\}$$

of the space Y. Also consider the finite (because f is a Λ-mapping) systems

$$\delta_i y = \{\tilde{\omega}_{i\alpha} \in \delta_i,\ f^{-1}y \cap \tilde{\omega}_{i\alpha} \neq \Lambda\}$$

and the open-and-closed in Y

* The system $\Sigma(y)$ is finitely additive if, together with a finite collection $U_1, \dots, U_s$ of its elements, it also contains the set $\bigcup_i U_i$ (the union of these elements).

(by virtue of the fact that f is a Λ-mapping) the sets

$$W_y^i = f^{\#}\delta_i y \cap \bigcap_{\tilde{\omega}_{ia} \in \delta_i y} f\omega_{ia}. \tag{1}$$

Just as in Theorem 1, we assert that $\tau_i = \{W_y^i,\ y \in Y\}$ is a disjoint open-and-closed cover, inscribed in the open-and-closed cover $f\delta_i$. Moreover, note that

$$W_{y_1}^i = W_{y_2}^i, \quad \text{if } y_1 \in W_{y_2}^i \text{ or } y_2 \in W_{y_1}^i. \tag{2}$$

Now consider an arbitrary $W_{y_0}^i \in \tau_i$ and

$$\omega_i y_0 = \bigcup_{\tilde{\omega}_{ia} \in \delta_i y_0} \omega_{ia}. \tag{3}$$

Also consider the system $\Sigma(\omega_i y_0)$, consisting of sets that are finite unions of sets from $\omega_i y_0$. Note that $\omega_i y_0$ is countable; hence the system $\Sigma(\omega_i y_0)$ will also be countable. Finally, consider

$$\xi(W^i_{y_0}) = \{W^i_{y_0} \cap I[fV],\ V \in \Sigma(\omega_i y_0)\}. \tag{4}$$

The system $\xi(W^i_{y_0})$ is also countable. From (2) and Lemma 1 there follows the equality

$$\tilde{\xi}(W^i_{y_0}) = W^i_{y_0}. \tag{5}$$

Now consider the systems

$$\xi_i = \bigcup_{W^i \in \tau_i} \xi(W^i). \tag{6}$$

From equality (5), the disjointness of the open-and-closed cover τ_i, and the countability of $\xi(W^i)$, it follows that ξ_i is a star-countable open cover of the space Y. Let now $i_1 i_2, \dots, i_k$ be some finite sequence of natural numbers. For each point $y_0 \in Y$, consider the systems $(\omega_{i_1} y_0, \dots, \omega_{i_k} y_0)$ and the system $\Sigma(\omega_{i_1} y_0 \cup \dots \cup \omega_{i_k} y_0)$, consisting of all possible finite unions of sets of the system $\omega_{i_1} y_0 \cup \omega_{i_2} y_0 \cup \dots \cup \omega_{i_k} y_0$. Also consider the sets $W^{i_1}_{y_0}, \dots, W^{i_k}_{y_0}$. Denote

$$\xi(W^{i_1} y_0, \dots, W^{i_k} y_0) = \{W^{i_1} y_0 \cap \dots \cap W^{i_k} y_0 \cap I[fV],\ V \in \Sigma(\omega_{i_1} y_0 \cup \dots \cup \omega_{i_k} y_0)\}. \tag{7}$$

From equality (5) and the inclusion

$$I[f(V^{i_1} \cup \dots \cup V^{i_k})] \supset I[fV^{i_1}] \cap \dots \cap I[fV^{i_k}] \tag{8}$$

there follows the equality

$$\tilde{\xi}(W^{i_1} y_0, \dots, W^{i_k} y_0) = W^{i_1} y_0 \cap \dots \cap W^{i_k} y_0. \tag{9}$$

Denote

$$\xi_{i_1 i_2 \dots i_k} = \bigcup \{\xi(W^{i_1}, \dots, W^{i_k}),\ W^{i_1} \in \tau_{i_1}, \dots, W^{i_k} \in \tau_{i_k}\}. \tag{10}$$

From equality (9), the countability of the systems $\xi(W^{i_1}, \dots, W^{i_k})$, $W^{i_1} \in \tau_{i_1}, \dots, W^{i_k} \in \tau_{i_k}$, and the disjointness of the cover $\tau_{i_1 i_2 \dots i_k} = \tau_{i_1} \wedge \dots \wedge \tau_{i_k}$, there follows the star-countability of the system $\xi_{i_1 i_2 \dots i_k}$.

Consider

$$\xi = \bigcup_{i_1 i_2 \dots i_k} \xi_{i_1 i_2 \dots i_k}. \tag{11}$$

The system ξ consists of open sets and is the union of a countable number of star-countable systems $\xi_{i_1, i_2 \dots i_k}$. We shall prove that ξ is a base of the space Y. Let $y_0 \in Y$ and let the neighborhood Oy_0 be arbitrary. Consider

a countable system

$$\omega y_0 = \bigcup_{i=1}^{\infty} \omega_i y_0 \tag{12}$$

and the countable system $\Sigma(\omega y_0)$, consisting of all possible unions of a finite number of elements from ωy_0. By Lemma 2, the system

$$I[f\Sigma(\omega y_0)] = \{I[fV],\ V \in \Sigma(\omega y_0)\} \tag{13}$$

forms a base at the point y_0. Therefore there will be such a $V^* \in \Sigma(\omega y_0)$ that

$$y_0 \in I[fV^*] \subseteq Oy_0. \tag{14}$$

But the equality

$$\Sigma(\omega y_0) = \bigcup_{i_1 i_2 \dots i_k} \Sigma(\omega_{i_1} y_0 \cup \omega_{i_2} y_0 \cup \dots \cup \omega_{i_k} y_0) \tag{15}$$

is valid.

Consequently, there exists a sequence $i_1, \dots, i_s$ such that

$$V^* \in \Sigma(\omega_{i_1} y_0 \cup \omega_{i_2} y_0 \cup \dots \cup \omega_{i_s} y_0). \tag{16}$$

Then $y_0 \in W^{i_1} y_0 \cap \dots \cap W^{i_s} y_0 \cap I[fV^*] = O^* y_0 \neq \Lambda$ and $O^* y_0 \subseteq Oy_0$ (14). Moreover, by the definition of the system $\xi(W^{i_1} y_0, \dots, W^{i_s} y_0)$ (7), necessarily $O^* y_0 \in \xi(W^{i_1} y_0, \dots, W^{i_s} y_0) \subseteq \xi_{i_1 i_2 \dots i_s} \subseteq \xi$, whereby it has been proved that ξ is a base of the space Y; here σ is star-countable, whence (by virtue of Proposition 2) the strong metrizability of the space Y follows. Everything is proved.

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Note: Figure translations are in progress. See original paper for figures.