ON TOPOLOGICAL GROUPS

MATHEMATICS

1969

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Abstract

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UDC 513.83

MATHEMATICS

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ON TOPOLOGICAL GROUPS

(Presented by Academician P. S. Aleksandrov on 10 I 1969)

All spaces considered are completely regular. We assume $\dim X = \dim \beta X$, where βX is the maximal bicompact extension of X.

I. The final solution of Hilbert' s fifth problem (see ($^{1-3}$)) can be formulated as follows:

A topological group is a Lie group if and only if it is locally bicompact, locally connected, and finite-dimensional.

In the class of normal spaces the property of countable compactness is considerably weaker than the property of bicompactness, and in the class of completely regular spaces an even weaker property is that of pseudocompactness (every continuous function is bounded).

Theorem 1. *A topological group G is a Lie group if and only if it is locally pseudocompact (in particular, countably compact), locally connected, and finite-dimensional.*

Proof. In one direction the assertion is obvious. Suppose that the group G is locally pseudocompact, locally connected, and $\dim G = \dim \beta G < \infty$. The completion $\overline{G}$ of the group G with respect to its two-sided uniformity U is a locally bicompact group, since the group G is locally completely bounded ((^{4}), Theorem 1.1). The group $\overline{G}$, obviously, is locally connected and, if we prove that $\dim \overline{G} < \infty$, then we shall prove that $\overline{G}$ is a Lie group, and then (by the metrizability of $\overline{G}$ and G) $\overline{G} \equiv G$. Thus it remains to show that the following is true.

Lemma 1. $\dim \overline{G} < \infty$, if $\dim G < \infty$.*

Proof. Take in $\overline{G}$ a neighborhood of the identity V, the closure of which in $\overline{G}$ is pseudocompact. Then the set $\overline{V} = \overline{G} \setminus [G \setminus V]_{\overline{G}}$ is a neighborhood of the identity in $\overline{G}$, whose closure in $\overline{G}$ is bicompact (since $\overline{V} \subseteq [V]_{\overline{G}}$). Since every locally bicompact group contains an open projective-Lie subgroup (^{3}), without restricting the generality of the argument the group $\overline{G}$ may be assumed to be projective-Lie. Then in $\overline{V}$ there is such a bicompact normal divisor N that $H = \overline{G}/N$ is a Lie group. The set

$$N_1 = N \cap [V]_G \tag{1}$$

is pseudocompact and

$$[N_1]_{\overline{G}} = N. \tag{2}$$

Let us first prove (2). If $g_0 \in N \setminus [N_1]_{\overline{G}}$, then the layer $[[V]_G]_{\overline{G}} \setminus [V]_G = [V]_{\overline{G}} \setminus [V]_G$ contains a nonempty bicompact set $B \ni g_0$ of type G_δ in $[V]_{\overline{G}}$ (N has type G_δ in $[V]_{\overline{G}}$, and the bicompacts g_0 and $[N_1]_{\overline{G}}$ have disjoint neighborhoods), but this contradicts the pseudocompactness of $[V]_G$. Thus, $[N_1]_{\overline{G}} = N$, consequently, $\overline{N}_1 \equiv N$.** We show that every nonempty set Γ of type G_δ in N intersects N_1. Let $\Gamma \cap N_1 = \Lambda$. Then, since the set Γ has type G_δ also in $[V]_G$, in Γ there is a bicompact set of type G_δ in $[V]_{\overline{G}}$, not intersecting $[V]_G$, and this cannot be by virtue of pseudocompactness $[V]_G$. By Theorem 1.2 (items (a) and (d)) from ([4]), the group N_1 is pseudocompact; consequently, $\overline{N}_1 \equiv N = \beta N_1$ (([4]), Theorem 4.1), whence $\dim N = \dim N_1$. Since already $[N_1]_{\beta\overline{G}} = \beta N_1$, a fortiori $[N_1]_{\beta G} = \beta N_1$, whence $\dim N = \dim \beta N_1 \leq \dim \beta G < \infty$. But then $\dim \overline{G} \leq \dim N + \dim \overline{G}/N < \infty$ (5). Lemma 1 (and hence Theorem 1) is proved.

II. **Definition 1.** A uniformly continuous mapping p of a uniform space G with uniformity U onto a uniform space X with uniformity U_p will be called, respectively: a) **twice uniformly continuous**, if for every uniform cover ω of the space G there exists a uniform cover $v = \{O_\alpha,\ \alpha \in \mathfrak{A}\}$ of the space X such that, for every α, the inverse image $p^{-1}O_\alpha$ can be covered by a finite system of elements of the cover ω; b) **uniformly perfect**, if the mapping p is closed and all sets $p^{-1}x,\ x \in X$, are complete with respect to the uniformity U.

Lemma 2. *If a mapping $p: G \to X$ is twice uniformly continuous, and a system of closed sets $F_\alpha \subseteq X,\ \alpha \in \mathfrak{A}$, is centered (i.e. the intersection of the elements of every finite subsystem of this system is nonempty) and fine with respect to the uniformity U_p, then the maximal centered system μ of closed sets $\Phi_\beta \subseteq G,\ \beta \in \mathfrak{B}$, containing as a subsystem the system $\{p^{-1}F_\alpha,\ \alpha \in \mathfrak{A}\}$, is fine with respect to the uniformity U.*

Proof. Take a uniform cover $\omega = \{V_\gamma,\ \gamma \in \Gamma\}$ of the space G. In accordance with Definition 1, take a uniform cover $v = \{O_\delta,\ \delta \in D\}$ of the space X. Then for some α_0 the set F_{α_0} is contained in some element O_{δ_0} of the cover v.

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* M. Choban independently proved the stronger assertion: $\dim \overline{G} = \dim G$.

** N_1 denotes the completion of N_1.

Let the inverse image $p^{-1}O_{\delta_0}$ be covered by the elements V_{γ_i}, $i = 1, \dots, s$, of the cover ω. The intersection of the sets $\Psi_i = p^{-1}F_{\alpha_0} \setminus V_{\gamma_i}$, $i = 1, \dots, s$, is empty; therefore at least one of them, for example Ψ_1, does not belong to the system μ. Then $\Psi_1 \cap \Phi_{\beta_0} = \Lambda$ for at least one $\beta_0 \in \mathfrak{B}$; consequently, $p^{-1}F_{\alpha_0} \cap \Phi_{\beta_0} \subseteq V_{\gamma_1}$, but $p^{-1}F_{\alpha_0} \cap \Phi_{\beta_0} \in \mu$. The lemma is proved.

From Lemma 1 it follows that

Proposition 1. *If a mapping $p : G \to X$ is twice uniformly continuous, then the completeness of the uniform space G implies the completeness of the uniform space X.*

Proposition 2. *If a mapping $p : G \to X$ is uniformly perfect, then the completeness of the uniform space X implies the completeness of the uniform space G.* *

Corollary 1. *If a mapping $p : G \to X$ of uniform spaces G and X is twice uniform and uniformly perfect, then for the completeness of each of the spaces G and X it is sufficient (and necessary) that one of them be complete.*

III. Let a group G be given, and in it a bicompact subgroup H. By Σ_l, Σ_r, and Σ denote, respectively, the set of all covers $\omega_{vl} = \{gV\}$, $\omega_{vr} = \{Vg\}$, and $\omega_v = \{gV \cap Vg\}$, $g \in G$, of the group G, where V is an arbitrary neighborhood of the identity of the group. As is known, the systems Σ_l, Σ_r, and Σ are bases of the uniformities L, R, and U on the group G.

By X_l, respectively X_r, denote the space of left, respectively right, cosets of the group G modulo the subgroup H. The natural projections $p_l : G \to X_l$ and $p_r : G \to X_r$ are not only open, but also, by virtue of the bicompactness of H, perfect (i.e. closed and bicompact). The mappings p_l and p_r determine a mapping $p_l \times p_r : G \to X_l \times X_r$. Denote the set $p_l \times p_r(G)$ by X, and denote the mapping $p_l \times p_r : G \to X$ by p. Obviously, the mapping p is perfect. It is not hard to show that, by virtue of the perfectness of the mappings p_l and p_r, the set X is closed in $X_l \times X_r$.

It is clear that X is the quotient space of the space G with respect to the decomposition into bicompacts of the form $gH \cap Hg$, $g \in G$.

By $\omega_{pVl}, \omega_{pVr}$, and ω_{pV} we denote the open covers $\{p_l(gV) = p_l(gVH)\}$, $\{p_r(Vg) = p_r(HVg)\}$, and $\{p(gVH \cap HVg)\}$, $g \in G^*$ of the spaces X_l, X_r, and X, respectively.

Proposition 3. The systems Σ_p, Σ_{pl}, and Σ_{pr} of all possible covers $\omega_{pV}, \omega_{pVl}, \omega_{pVr}$, respectively, are bases of uniformities (we denote them by U_p, L_p, and R_p) on the spaces X, X_l, and X_r, respectively, compatible with the topology of these spaces.

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* For proximity spaces an analogous assertion was obtained earlier by V. Z. Polyakov in (^{10}).

Lemma 3. The mappings $p : G \to X,\ p_l : G \to X_l$, and $p_r : G \to X_r$ are twice uniformly continuous with respect to the uniformities U and $U_p,\ L$ and $L_p,\ R$ and R_p, respectively.

Corollary 2. a) If the group G is complete with respect to the uniformity U (complete in the sense of Raikov), then the space X is complete with respect to the uniformity U_p. b) If the group G is complete with respect to the uniformity L, and hence also R (complete in the sense of Weil), then the spaces X_l and X_r are complete with respect to the uniformities L_p and R_p, respectively.

Moreover: c) If a subset Γ of the group G is complete with respect to the uniformity U, respectively L, R, then the set $p\Gamma$, respectively $p_l\Gamma, p_r\Gamma$, is complete with respect to the uniformity U_p, respectively L_p, R_p.

Let now the group G also be given a subgroup $\Gamma \equiv H$ that is complete: a) with respect to the uniformity U, respectively b) with respect to the uniformities L and R, and hence closed.

By Y_l and Y_r we denote the spaces of left and right cosets of G modulo Γ, and by q_l and q_r the natural projections of G onto Y_l and Y_r, respectively. The mapping $q_l \times q_r : G \to q_l \times q_r(G) \subseteq Y_l \times Y_r$ will be denoted by q, and the set $q_l \times q_r(G)$ by Y. The natural projections X_l onto Y_l and X_r onto Y_r will be denoted by π_l and π_r, respectively. The naturally arising mapping of X onto Y will be denoted by π $(\pi(x_l, x_r) = (\pi_l x_l, \pi_r x_r))$.

Lemma 4. a) All sets $X_y = \pi^{-1}y,\ y \in Y$, are complete with respect to the uniformity U_p, respectively b) all sets $X_{y_l} = \pi_l^{-1}y_l,\ y_l \in Y_l$, and $X_{y_r} = \pi_r^{-1}y_r,\ y_r \in Y_r$, are complete with respect to the uniformity L_p and R_p, respectively.

Proposition 4. If the group H has countable character in the group Γ, then: a) for each $y \in Y$, respectively b) $y_l \in Y_l$ and $y_r \in Y_r$, the uniformity U_p on X_y, respectively L_p on X_{y_l} and R_p on X_{y_r}, is metrizable and complete.**

Theorem 2. If a subgroup Γ of the group G is complete in the sense of Weil and almost metrizable (7), then for any paracompact A contained in the quotient space Y_l (respectively in Y_r) there exists in G a closed subset B that is mapped perfectly onto A by the projection q_l (respectively q_r). If moreover $\dim A = 0$, then the mapping $q_l : B \to A$ may be assumed to be a homeomorphism. If $\Delta X \leq n$, (8), then the mapping $q_l : B \to A$ may be assumed to be at most $(n+1)$-to-one.

Corollary 3. If a subgroup Γ of the group G is complete in the sense of Weil and almost metrizable, and the quotient space Y_l (respectively Y_r) is paracompact and $\dim Y_l = 0$, then the space of the group G is homeomorphic to the product $Y_l \times \Gamma$.

Theorem 3. If a normal divisor Γ of the group G is complete in the sense of Čech, then for any paracompact $A \subseteq G/\Gamma$ there exists in the group G a set B that is mapped perfectly onto A by the projection q;

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* It is not difficult to show that the set $gVH \cap HVg$ is the preimage of the set $p(gVH \cap HVg)$ under the mapping p, since
$p(gVH \cap HVg) = p(gVH) \cap p(HVg)$.

** Thus, all sets $p^{-1}X_y$ are complete in the sense of Čech, and for $\Gamma \equiv G$ we obtain M. Choban' s result on the Čech-completeness of a Raikov-complete almost metrizable group.

if $\dim A = 0$, then the mapping $q : B \to A$ may be regarded as a homeomorphism; if, however, $\Delta A \leq n$, then this mapping may be regarded as no more than $(n + 1)$-fold.

Corollary 4. If a (quasi-)component Γ of a group G is complete in the sense of Čech, then for any locally bicompact set $A \subseteq G/\Gamma$ the mapping $q : q^{-1}A \to A$ is a locally trivial fibration; if the set A is also paracompact, for example bicompact, then the mapping $q : q^{-1}A \to A$ is a trivial fibration.

Corollary 5. If a group G is strongly paracompact, its (quasi-)component Γ is complete in the sense of Čech, and $\operatorname{ind} G/\Gamma = 0$, then the space of the group G is homeomorphic to the product of the spaces Γ and G/Γ.

This corollary generalizes Mostert' s theorem stating that the space of a locally bicompact group G is homeomorphic to the product of its component Γ and the quotient space G/Γ (^{6}).

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Received
20 XII 1968

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Note: Figure translations are in progress. See original paper for figures.

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