ON THE APPLICATION OF NEWTON’ S METHOD TO THE DETERMINATION OF EIGENVALUES OF \(\lambda\)-MATRICES

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 518:517

MATHEMATICS

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ON THE APPLICATION OF NEWTON' S METHOD TO THE DETERMINATION OF EIGENVALUES OF λ-MATRICES

(Presented by Academician L. V. Kantorovich, 24 II 1969)

Let $D(\lambda)$ be a given square matrix of order n, whose elements are regular functions of the scalar parameter λ in some prescribed domain. By an eigenvalue μ of the matrix $D(\lambda)$ is meant a root of the equation

$$\det D(\lambda) = 0. \tag{1}$$

Vectors U and V satisfying the equations $D(\mu)U = 0$ and $V^T D(\mu) = 0$ are called, respectively, the right and left eigenvectors corresponding to the eigenvalue μ.

Below we describe a process that makes it possible, for determining an isolated eigenvalue of the matrix $D(\lambda)$, to apply Newton' s method without expanding the determinant $D(\lambda)$. The process is based on orthogonal transformations, which make it possible to pass from solving equation (1) to solving the equation

$$f(\lambda) = 0, \tag{2}$$

whose roots coincide with the distinct roots of equation (1). The left-hand side of equation (2) is not specified explicitly, but a rule is given for determining $f(\lambda)$ and $f(\lambda)/f'(\lambda)$ for fixed λ. The process makes it possible to construct left and right eigenvectors of $D(\lambda)$ corresponding to the eigenvalue found.

It is known that the matrix $D(\lambda)$, for any fixed value $\lambda = \lambda_k$, can be represented in the form

$$\theta_k D(\lambda_k) = L_k Q_k^T, \tag{3}$$

where L_k is a lower triangular matrix whose diagonal elements are arranged in the order

$$l_1^{(k)} \geq l_2^{(k)} \geq \cdots \geq l_{n-1}^{(k)} \geq |l_n^{(k)}|;$$

Q_k is an orthogonal matrix ($\det Q_k = 1$), and θ_k is the resulting permutation matrix.

The decomposition (3) can be obtained, for example, by a normalized process ([1]), which reduces to multiplying the matrix $D(\lambda_k)$ on the right by a chain of elementary rotation or reflection matrices that make it possible to introduce zeros in the required positions and, only if necessary, to permute rows of the matrix being processed.

Let $f(\lambda)$ denote the function of λ whose value, for any fixed λ, coincides with $(-1)^{m_k} l_n^{(k)}$, where m_k is the number of row permutations of the matrix $D(\lambda_k)$ made in the course of reducing $D(\lambda_k)$ to the form (3). From the obvious relation

$$\det D(\lambda_k) = \det L_k \cdot \det \theta_k = l_1^{(k)} \cdots l_n^{(k)} \cdot (-1)^{m_k} = l_1^{(k)} \cdots l_{n-1}^{(k)} f(\lambda_k),$$

where $|f(\lambda_k)| = |l_n^{(k)}|$ is the smallest among the numbers $l_1^{(k)}, l_2^{(k)}, \dots, l_{n-1}^{(k)}, |l_n^{(k)}|$, it follows that the distinct roots of equation (1) coincide with the roots of equation (2).

Let an approximation λ_0 to the eigenvalue be known, sufficient for applying Newton' s method. We shall show how to find the correction $\Delta_k =$

$$= f(\lambda_k)/f'(\lambda_k)$$

for constructing subsequent approximations $\lambda_1, \lambda_2, \dots$ by Newton' s method. To this end we differentiate the equality

$$\theta(\lambda) D(\lambda) = L(\lambda) Q^T(\lambda).$$

We have

$$\theta'(\lambda) D(\lambda) + \theta(\lambda) D'(\lambda) = L'(\lambda) Q^T(\lambda) + L(\lambda)(Q^T(\lambda))'. \tag{4}$$

We shall assume that λ belongs to a neighborhood of the root being determined in which the matrix $\theta(\lambda)$ has already stabilized, so that $\theta'(\lambda) = 0$. Then, multiplying (4) on the left and on the right respectively by $L^{-1}(\lambda)$ and $Q(\lambda)$, we obtain

$$L^{-1}(\lambda)\theta(\lambda) D'(\lambda) Q(\lambda) = L^{-1}(\lambda) L'(\lambda) + (Q^T(\lambda))' Q(\lambda). \tag{5}$$

This equality is valid for any λ from the neighborhood under consideration, distinct from an eigenvalue of the matrix $D(\lambda)$. But the matrix $(Q^T(\lambda))' Q(\lambda)$ is skew-symmetric for any λ, as is easily seen by differentiating the identity $Q^T(\lambda) Q(\lambda) = E$. In view of what has been said, from (5) we have

$$\operatorname{diag}[L^{-1}(\lambda)\theta(\lambda) D'(\lambda) Q(\lambda)] = \operatorname{diag}[L^{-1}(\lambda) L'(\lambda)]. \tag{6}$$

The matrix $L'(\lambda)$ is lower triangular, so that, equating the last elements in (6), we find $l_n^{-1} l_n' = x_n$, where x_n denotes the last component of the solution X of the system

$$LX = \theta D' q,$$

q being the last column of the matrix Q.

Thus,

$$\Delta_k = f(\lambda_k)/f'(\lambda_k) = l_n^{(k)}/(l_n^{(k)})' = 1/x_n^{(k)},$$

so that Newton' s method formula is written as

$$\lambda_{k+1} = \lambda_k - 1/x_n^{(k)}.$$

Suppose that the eigenvalue $\mu \approx \lambda_k$ has been found with sufficient accuracy. Then the last column of the matrix Q_k in the decomposition (3) will be an approximation to the right eigenvector U of the matrix $D(\lambda)$ corresponding to μ. The construction of the left eigenvector corresponding to the eigenvalue μ is carried out analogously, if one obtains the decomposition (3) for the matrix $D^T(\lambda_k)$.

If the eigenvalue μ corresponds to r linearly independent eigenvectors of $D(\lambda)$, then the last r diagonal elements of the matrix L in the decomposition (3) obtained for $\lambda = \mu$ will be zero ([2]): $l_n = ... = l_{n-r+1} = 0$, $l_{n-r} \neq 0$. In this case the last r columns of the orthogonal matrix Q will give all linearly independent eigenvectors of the matrix $D(\lambda)$ corresponding to μ.

The described process may be applied to refining an isolated eigenvalue of a square matrix A. In this case $D(\lambda) = A - \lambda E$.

Remark. As already noted, determining the different roots of equation (1) is equivalent to solving equation (2). We do not know the explicit dependence $f(\lambda)$, but for any fixed λ we can find the value $f(\lambda)$. It should be assumed that, for solving (2), one can also apply other methods for determining zeros of functions, for example methods based on the idea of interpolation. It may be that this will require additional investigations of the function $f(\lambda)$.

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Received
11 II 1969

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Note: Figure translations are in progress. See original paper for figures.