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Abstract

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MATHEMATICS

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ON THE BOUNDEDNESS OF ONE CLASS OF SINGULAR INTEGRAL OPERATORS

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Singular integral operators, considered by S. G. Mikhlin ([1,2]) and A. P. Calderón and A. Zygmund ([3]), play an increasingly important role in the theory of differential equations with partial derivatives. With their help, general existence and uniqueness theorems have been proved. In the present note we indicate a class of singular integrals whose kernels have a singularity lying on a hypersurface and which are closely connected with the generalized shift operator ([4]).

1. Let E^+_{n+2} be the Euclidean half-space of points (s, z_1, z_2) of dimension $n+2$ $(s=(s_1,\dots,s_n), z_2 \geq 0)$.

Consider a singular integral of the form

$$v(x,y) = C_k \int_{E^+_{n+2}} \frac{f(\tilde{\theta})\varphi\left(s, \sqrt{z_1^2+z_2^2}\right) z_2^{k-1}\, ds\, dz_1\, dz_2}{\left[\sum (x_i - s_i)^2 + (y-z_1)^2 + z_2^2\right]^{(n+k+1)/2}}, \tag{1}$$

where $\tilde{\theta} = (\tilde{\theta}_1, \dots, \tilde{\theta}_{n+1})$,
$\tilde{\theta}_i = (x_i - s_i)/r$, $\tilde{\theta}_{n+1} = \sqrt{(y-z_1)^2 + z_2^2}/r$,
$r^2 = \sum (x_i - s_i)^2 + (y-z_1)^2 + z_2^2$,
$C_k = \Gamma(k+1/2)/\Gamma(1/2)\Gamma(k)$. The number k is positive and the coordinate y is nonnegative.

By the definition of the singular integral we set

$$v(x,y) = \lim_{\varepsilon\to 0} C_k \int_{r>\varepsilon} \frac{f(\tilde{\theta})\varphi\left(s, \sqrt{z_1^2+z_2^2}\right)}{r^{\,n+k+1}} z_2^{k-1}\, ds\, dz_1\, dz_2. \tag{2}$$

We shall assume that the density φ and the characteristic f satisfy, as usual, the following conditions: 1) in any bounded part of the half-space E^+_{n+2} the function φ satisfies the Hölder condition with exponent $\alpha > 0$; 2) at infinity $\varphi(x,y) = O(r^{-\beta})$, where $r^2 = |x|^2 + y^2$, $\beta > 0$; 3) the characteristic $f(\theta)$ is bounded and continuous. Under these conditions, for the singular integral (1) to exist it is necessary and sufficient that the condition

$$\int_{\Omega_{n+2}^+} f(\tilde{\theta}) z_2^{k-1}\, d\omega_{n+2} = 0, \tag{3}$$

be satisfied, where Ω_{n+2}^+ is the unit hemisphere in E_{n+2}^+ with center at the point $(x, y, 0)$.

The singular integral (1) can also be written in another form. Put $z\cos\gamma = z_1$, $z\sin\gamma = z_2$; then

$$v(x,y) = C_k \int_{E_{n+1}^+} \int_0^\pi \frac{f(\bar{\theta})}{\tilde{r}^{\,n+k+1}} \varphi(s,z) z^k \sin^{k-1}\gamma\, ds\, dz\, d\gamma, \tag{4}$$

where $\bar{\theta}_i = (x_i - s_i)/\tilde{r}$,
$\bar{\theta}_{n+1} = \sqrt{y^2 + z^2 - 2yz\cos\gamma}/\tilde{r}$,
$\tilde{r} = \left(|x-s|^2 + y^2 + z^2 - 2yz\cos\gamma\right)^{1/2}$, and E_{n+1}^+ is the $(n+1)$-dimensional half-space

$(s, z \geqslant 0)$. Taking into account the explicit expression of the generalized shift operator (4),

$$T_{x,y}^{s,z} f(x,y) = \frac{\Gamma(k+1/2)}{\Gamma(1/2)\Gamma(k)} \int_0^\pi f\left(x-s, \sqrt{y^2+z^2-2yz\cos\gamma}\right) \sin^{k-1}\gamma\, d\gamma, \tag{5}$$

we now write the integral (4) in the form

$$v(x,y) = \int_{E_{n+1}^+} T_{x,y}^{s,z} \frac{f(\theta)}{r^{n+k+1}} \varphi(s,z) z^k\, ds\, dz, \tag{6}$$

where $\theta_i = x_i/r,\ \theta_{n+1} = y/r,\ r^2 = |x|^2 + y^2$.

Since φ is summable over any finite domain, and the operator $T_{x,y}^{s,z}$ is formally self-adjoint (4), the integral (6) can also be written in the form

$$v(x,y) = \int_{E_{n+1}^+} \frac{f(\theta)}{r^{n+k+1}} T_{x,y}^{s,z} \varphi(x,y) z^k\, ds\, dz, \tag{7}$$

where $\theta_i = s_i/r,\ \theta_{n+1} = z/r,\ r^2 = |s|^2 + z^2$.

Let us note that if in formula (2) one performs the ordinary shift in the variables s and z_1, then we obtain

$$v(x,y) = C_k \lim_{\varepsilon\to 0} \int_{r>\varepsilon} \frac{f(\theta)}{r^{n+k+1}} \varphi\left(x-s, \sqrt{(y-z_1)^2+z_2^2}\right) z_2^{k-1}\, ds\, dz_1\, dz_2,$$

where $\theta_i = s_i/r,\ \theta_{n+1} = \sqrt{z_1^2+z_2^2}/r,\quad r^2 = |s|^2+z_1^2+z_2^2$. The substitution $z_1 = z\cos\gamma,\ z_2 = z\sin\gamma$ transforms this integral, taking (5) into account, to the form

$$v(x,y) = \lim_{\varepsilon\to 0}\int_{r>\varepsilon} \frac{f(\theta)}{r^{n+k+1}} T_{x,y}^{s,z}\varphi(x,y) z^k\, ds\, dz, \tag{8}$$

where $\theta_i,\ \theta_{n+1}$, and r have the same meaning as in (7).

2. For the singular integral considered, theorems of the type of the theorems of I. I. Privalov hold (see, for example, ([2],[5])).

For $0<\alpha<1$ introduce the seminorms

$$[f]_\alpha = \sup\frac{|f(x,y)-f(s,z)|}{[|x-s|^2+(y-z)^2]^{\alpha/2}}, \qquad [u]_\alpha = \sup\frac{|u(P)-u(Q)|}{|P-Q|^\alpha},$$

where $P = P(x,y,t),\ Q = Q(c,z,t)$. Denote by $\mathcal{L}_{p,k}$ the set of functions summable in the half-space E_{n+1}^+ to degree $p \geqslant 1$ with weight y^k $(y\geqslant 0,\ k>0)$.

Theorem 1. Let

$$K(x,y,t) = \frac{\Omega(x/r, y/r)}{(r^2+t^2)^{(n+k+1)/2}}, \qquad r^2 = |x|^2+y^2, \tag{9}$$

where $\Omega(x,y)$ satisfies on $|x|^2+y^2=1$ the Hölder condition with exponent α' and constant χ such that $|\Omega(x,y)|\leqslant\chi$, and, moreover,

$$\int_{r=1}\Omega(x,y)y^k\, dx\, dy = 0. \tag{10}$$

Let $f(x,y)\in\mathcal{L}_{p,k}$ and $[f]_\alpha<\infty$.

Then for the integral operator

$$u(x,y,t) = \int_{E_{n+1}^+} T_{x,y}^{s,z} K(x,y,t) f(s,z) z^k\, ds\, dz \tag{11}$$

the estimate

$$[u]_\alpha \leqslant C\chi[f]_\alpha$$

is valid, where the constant C depends only on α, α', and $n+k+1$.

The following theorem is also true, following directly from Theorem 1:

Theorem 2. Let the function $f(x,y)$ belong to $\mathcal{L}_{p,k}$ and satisfy the Hölder condition

$$|f(x,y) - f(s,z)| \leq M\left(|x-s|^2 + (y-z)^2\right)^{\alpha/2}.$$

Let

$$u(x,y) = \lim_{\varepsilon \to 0} \int_{r>\varepsilon} \frac{\Omega(s/r, z/r)}{r^{n+k+1}}\, T_{x,y}^{s,z} f(x,y)\, z^k\, ds\, dz,$$

where $\Omega(x,y)$ satisfies, on $|x|^2 + y^2 = 1$, a Hölder condition with exponent α' and with constant χ such that $|\Omega| \leq \chi$, and condition (10). Then

$$|u(x,y) - u(s,z)| \leq C\chi M\left(|x-s|^2 + (y-z)^2\right)^{\alpha/2},$$

where C depends only on α, α', and $n+k+1$.

3. In conclusion we note that theorems of the type of the well-known theorems of A. P. Calderon and A. Zygmund (see, for example, $(3,5)$) on boundedness also hold for our singular integral operator in the space $\mathcal{L}_{p,k}$.

Theorem 3. Let an operator (11) be given, where the kernel $K(x,y,t)$ is defined by formula (9) and the angular function $\Omega(x,y)$ satisfies the Hölder condition and condition (10). Then, if $f(x,y) \in \mathcal{L}_{p,k}$, $p > 1$, then for any $t > 0$, $u(x,y,t)$, as a function of (x,y), belongs to $\mathcal{L}_{p,k}$, and the inequality

$$\|u(x,y,t)\|_{\mathcal{L}_{p,k}} \leq C\|f\|_{\mathcal{L}_{p,k}},$$

holds, where the constant C does not depend on f or t.

Theorem 4. Let the kernel $K(x,y)$ be given by the formula

$$K(x,y) = \frac{\Omega(x/r, y/r)}{r^{n+k+1}}, \qquad r^2 = |x|^2 + y^2,$$

where the function $\Omega(x,y)$ satisfies condition (10) and the condition

$$\left\{\int_{r=1} |\Omega(x,y)|^q y^k\, dx\, dy\right\}^{1/q} = R_q < \infty \qquad (q > 1).$$

Then for any function $f(x,y) \in \mathcal{L}_{p,k}$ there exists, in the sense of convergence in $\mathcal{L}_{p,k}$, the limit

$$u(x,y) = \lim_{\varepsilon\to 0} \int_{r>\varepsilon} K(s,z)\, T^{s,z}_{x,y} f(x,y)\, z^k\, ds\, dz, \tag{12}$$

belonging to $\mathcal{L}_{p,k}$ and satisfying the inequality

$$\|u\|_{\mathcal{L}_{p,k}} \le C R_q \|f\|_{\mathcal{L}_{p,k}},$$

where C does not depend on f.

We note that in the course of the proof of Theorems 3 and 4 it becomes clear that, in the sense of convergence in $\mathcal{L}_{p,k}$, there exists the limit $u(x,y,t)$ as $t \to 0$, equal to $u(x,y)$, belonging to $\mathcal{L}_{p,k}$ and being a certain regularization of the divergent integral

$$\int_{E^+_{n+1}} T^{s,z}_{x,y} K(x,y) f(s,z) z^k\, ds\, dz. \tag{13}$$

Formula (12), however, gives another regularization of the divergent integral (13). It is shown that this regularization leads to the very same function $u(x,y)$.

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CITED LITERATURE

1. S. G. Mikhlin, *UMN*, **3**, 25 (1948).

2. S. G. Mikhlin, *Multidimensional Singular Integrals and Integral Equations*, Moscow, 1962.

3. A. P. Calderon, H. Zygmund, *Am. J. Math.*, **78**, No. 2, 310 (1956).

4. B. M. Levitan, *UMN*, **6**, No. 2 (42) (1951).

5. S. Agmon, A. Douglis, L. Nirenberg, *Estimates of Solutions of Elliptic Equations near the Boundary*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.