

# ON THE QUASI-EQUIVALENCE OF BASES IN AN INFINITE CENTER OF A HILBERT SCALE

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON THE QUASI-EQUIVALENCE OF BASES IN AN INFINITE CENTER OF A HILBERT SCALE

*(Presented by Academician L. V. Kantorovich, March 28, 1969)*

1°. Let  $\{x_k\}$  be an unconditional Schauder basis in a linear topological space  $E$ . It is obvious that the system

$$y_k = \lambda_k T(x_{n_k})$$

is an unconditional basis in  $E$  for any permutation of the natural sequence  $\{n_k\}$ , any sequence of numbers  $\lambda_k \neq 0$ , and any isomorphism  $T : E \rightarrow E$ . The bases  $\{x_k\}$  and  $\{y_k\}$  are called quasi-equivalent <sup>(1,2)</sup>.

A natural question arises: are all unconditional bases in  $E$  quasi-equivalent? This problem was solved positively for an entire class of nuclear countably normed spaces <sup>(1-4)</sup>. The assumption of nuclearity of the space was removed in <sup>(5)</sup> for the case of so-called finite centers of Hilbert scales<sup>\*\*</sup>. In the present paper the quasi-equivalence of all unconditional bases in infinite centers of completely continuous Hilbert scales is established. The proof of quasi-equivalence in infinite centers differs qualitatively from the arguments of <sup>(5)</sup>. This is due to the fact that Lemma 3 of <sup>(5)</sup> has no analogue for infinite centers.

2°. Every positive self-adjoint bounded operator  $A$  in a Hilbert space  $H$  generates a Hilbert scale  $\{H_\alpha\}$  of Hilbert spaces  $H_\alpha$  with scalar products

$$(x, y)_{H_\alpha} = (A^{-\alpha}x, A^{-\alpha}y),$$

where  $(x, y)$  is the scalar product in  $H = H_0$  <sup>(1,6)</sup>. We shall call the scale  $\{H_\alpha\}$  completely continuous if the operator  $A$  is completely continuous. In the latter case one may assume that  $H_\alpha = l_2(a_k^\alpha)$ , where  $a_k = 1/\lambda_k$ ,  $\lambda_k$  are the eigenvalues of the operator  $A$ , arranged in decreasing order, i.e.  $a_k \uparrow \infty$ .

The countably Hilbert space

$$E_\alpha(a_k) \equiv \lim_{\lambda < \alpha} \text{pr } l_2(a_k^\lambda),$$

following (1), will be called the center of the scale  $\{H_\lambda\}$ —finite if  $\alpha < \infty$ , and infinite if  $\alpha = \infty$ . For  $x = (\xi_1, \xi_2, \dots, \xi_k, \dots) \in H_\lambda$  we shall denote

$$\|x\|_\lambda = \|x\|_{H_\lambda} = \left( \sum |\xi_k|^2 a_k^{2\lambda} \right)^{1/2}.$$

If  $H_0 \supset H_1$  are two Hilbert spaces with a (completely) continuous embedding, then there exists a (completely continuous) Hilbert scale  $\{G_\alpha\}$  such that  $G_0 = H_0$ ,  $G_1 = H_1$ . We shall say that  $\{G_\alpha\}$  is the scale generated by  $H_0$  and  $H_1$ , and use the notation

$$G_\alpha = (H_0)^{1-\alpha}(H_1)^\alpha.$$

We shall repeatedly use the following simple consequence of the known interpolation theorem (7).

\* This question was first posed by M. G. Haplanov for the space of analytic functions.

\*\* Here and below the terminology of paper (5) is used.

**Lemma** (cf. (9)). Let  $H_1 \supset H_2 \supset H_3 \supset H_4$  be a quadruple of Hilbert spaces with (complete) continuous embeddings. Then for any  $\alpha : 0 \leq \alpha \leq 1$  the following (complete) continuous embedding holds:

$$(H_1)^{1-\alpha}(H_3)^\alpha \supset (H_2)^{1-\alpha}(H_4)^\alpha.$$

3°. We shall prove the main result.

**Theorem.** In the space  $E = E_\infty(a_k)$ ,  $a_k \uparrow \infty$ , all unconditional bases are quasiequivalent.

Let  $\{x_k\}$  be an arbitrary unconditional basis in  $E$ . It is required to show that  $\{x_k\}$  is quasiequivalent to the principal basis  $\{e_k\}$ , consisting of the unit vectors. In other words, one has to find a permutation of the natural series  $\{n_k\}$  and a numerical sequence  $\sigma_k > 0$  such that the bases  $\{y_k = \sigma_k x_{n_k}\}$  and  $\{e_k\}$  are equivalent, i.e. the linear operator  $T$  defined by the formula

$$T \left( \sum \xi_k e_k \right) = \sum \xi_k y_k, \quad (1)$$

is an isomorphism of  $E$  onto itself.

Denote by  $\{x'_k\}$  the system biorthogonal to the basis  $\{x_k\}$ . By Lemma 1 (5), the system of Hilbert norms

$$|x|_\lambda^{(0)} = \left( \sum |x'_k(x)|^2 \|x_k\|_\lambda^2 \right)^{1/2}, \quad \lambda < \infty,$$

is equivalent to the original system of norms  $\{\|x\|_\lambda, \lambda < \infty\}$ . Denote by  $H_\lambda^{(0)}$  the Hilbert space obtained from  $E$  by completion with respect to the norm  $\|x\|_\lambda^{(0)}$ . From the equivalence of the systems of norms it follows that there exist numbers  $\mu_1 < \nu_1 < \nu_2 < \mu_2$ ,  $\lambda_1 < \lambda_2$  such that the embeddings

$$H_{\mu_1} \supset H_{\lambda_1}^{(0)} \supset H_{\nu_1} \supset H_{\nu_2} \supset H_{\lambda_2}^{(0)} \supset H_{\mu_2}$$

hold with continuous embeddings.

Take a permutation of the natural series  $\{n_k\}$  and a numerical sequence  $\sigma_k > 0$  such that the system  $y_k = \sigma_k x_{n_k}$  satisfies the conditions

$$\|y_k\|_{\lambda_1}^{(0)} = 1, \quad \|y_k\|_{\lambda_2}^{(0)} = b_k \uparrow \infty.$$

Below it will be shown that this permutation and renormalization are the desired ones, i.e. the operator (1) is an isomorphism of  $E$  onto itself.

Applying Lemma 2 <sup>(5)</sup> first to the quadruple of spaces

$$H_{\mu_1} \supset H_{\lambda_1}^{(0)} \supset H_{\lambda_2}^{(0)} \supset H_{\mu_2}$$

and the bases  $\{a_k^{-\mu_1} e_k\}$ ,  $\{y_k\}$ , and then to the quadruple of spaces

$$H_{\lambda_1}^{(0)} \supset H_{\nu_1} \supset H_{\nu_2} \supset H_{\lambda_2}^{(0)}$$

and the bases  $\{y_k\}$ ,  $\{a_k^{-\nu_1} e_k\}$ , we obtain the inequalities:

$$a_k^{\nu_2 - \nu_1} \prec b_k = \|y_k\|_{\lambda_2}^{(0)} \prec a_k^{\mu_2 - \mu_1}. \quad (2)$$

Here and in what follows the notation  $a_k \prec b_k$  means  $a_k = O(b_k)$ .

It follows from the lemma that for any  $\sigma : 0 \leq \sigma \leq 1$  the continuous embeddings

$$H_{\mu_1(1-\sigma) + \nu_2\sigma} \supset G_\sigma \supset H_{\nu_1(1-\sigma) + \mu_2\sigma}, \quad G_\sigma = (H_{\lambda_1}^{(0)})^{1-\sigma} (H_{\lambda_2}^{(0)})^\sigma$$

hold.

Hence, if one takes into account that  $\|y_k\|_{G_\sigma} = b_k^\sigma$ , the inequalities ( $\nu_1 < \nu < \nu_2$ )

$$b_k^{\sigma_2} \prec \|y_k\|_\nu \prec b_k^{\sigma_1}, \quad (3)$$

follow, where

$$\sigma_1 = \sigma_1(\nu) = (\nu - \nu_1)/(\mu_2 - \nu_1), \quad \sigma_2 = \sigma_2(\nu) = (\nu - \mu_1)/(\nu_2 - \mu_1).$$

Comparing inequalities (2) and (3), we obtain:

$$a_k^{\gamma_2} \prec \|y_k\|_\nu \prec a_k^{\gamma_1}, \quad (4)$$

where

$$\gamma_1 = \sigma_1(\mu_2 - \mu_1), \quad \gamma_2 = \sigma_2(\nu_2 - \nu_1).$$

Now take an arbitrary  $\lambda > \lambda_2^{(0)}$  and  $\tau = \tau(\lambda) \geq \nu_2$ ,  $\chi = \chi(\lambda)$  such that the continuous embeddings hold:

$$H_\tau \supset H_\lambda^{(0)} \supset H_\chi$$

and  $\tau(\lambda) \uparrow \infty$  as  $\lambda \uparrow \infty$ .

Let  $\{y_{n_k}\}$  be a rearrangement of the basis  $\{y_k\}$  in decreasing order of the norms in  $H_\lambda^{(0)}$ , i.e.,

$$\|y_{n_k}\|_\lambda^{(0)} = s_k \uparrow \infty.$$

Application of Lemma 2 (5), analogously to the preceding one, gives the inequalities:

$$a_k^{\tau-\nu_1} \prec \|y_{n_k}\|_\lambda^{(0)} = s_k \prec a_k^{\chi-\mu_1}. \quad (5)$$

As before, after applying the lemma of the present paper we obtain the estimates ( $\nu_1 < \nu < \tau$ ):

$$s_k^{\tilde{\sigma}_2} \prec \|y_{n_k}\|_\nu \prec s_k^{\tilde{\sigma}_1}, \quad (6)$$

where  $\tilde{\sigma}_1 = \tilde{\sigma}_1(\nu, \lambda) = (\nu - \nu_1)/(\chi - \nu_1)$ ,  
 $\tilde{\sigma}_2 = \tilde{\sigma}_2(\nu, \lambda) = (\nu - \mu_1)/(\tau - \mu_1)$ .

The following inequalities are the result of comparing inequalities (5) and (6):

$$a_k^{\tilde{\gamma}_2} \prec \|y_{n_k}\|_\nu \prec a_k^{\tilde{\gamma}_1}, \quad (7)$$

where  $\tilde{\gamma}_1 = \tilde{\sigma}_1(\chi - \mu_1)$ ,  $\tilde{\gamma}_2 = \tilde{\sigma}_2(\tau - \nu_1)$ .

Comparing inequalities (4) and (7), we obtain:

$$a_{n_k}^{\gamma_2/\tilde{\gamma}_1} \prec a_k \prec a_{n_k}^{\gamma_1/\tilde{\gamma}_2}.$$

Together with inequality (5) this gives

$$a_{n_k}^{\rho_2} \prec \|y_{n_k}\|_\lambda^{(0)} \prec a_{n_k}^{\rho_1}, \quad (8)$$

where

$$\rho_1 = \rho_1(\lambda) = \frac{\gamma_1}{\tilde{\gamma}_2}(\chi - \mu_1) = \frac{(\nu - \nu_1)(\mu_2 - \mu_1)(\tau - \mu_1)}{(\mu_2 - \nu_1)(\nu - \mu_1)(\tau - \nu_1)}(\chi - \mu_1),$$

$$\rho_2 = \rho_2(\lambda) = \frac{\gamma_2}{\tilde{\gamma}_1}(\tau - \nu_1) = \frac{(\nu - \mu_1)(\nu_2 - \nu_1)(\chi - \nu_1)}{(\nu_2 - \mu_1)(\nu - \nu_1)(\chi - \mu_1)}(\tau - \nu_1).$$

Obviously, as  $\lambda \uparrow \infty$ ,

$$\rho_2(\lambda) \uparrow \infty. \quad (9)$$

Since  $\{n_k\}$  is a rearrangement of the natural sequence, formula (8) may be rewritten in the form

$$C_2 a_k^{\rho_2} \leq \|y_k\|_\lambda^{(0)} \leq C_1 a_k^{\rho_1}, \quad C_1 = C_1(\lambda) < \infty, \quad C_2 = C_2(\lambda) > 0. \quad (10)$$

It is now easy to establish that the operator  $T$ , defined by formula (1), is an isomorphism.

I take this opportunity to thank B. S. Mityagin, who drew my attention to the fact that a number of essential points in the proof of Theorem 1 in paper <sup>(5)</sup> had been used earlier (for another purpose) in <sup>(2)</sup> (see the proof of Theorem 11).

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*Note: Figure translations are in progress. See original paper for figures.*

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