ON SUBLINEAR EXTENSIONS OF FUNCTIONALS

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Abstract

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MATHEMATICS

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ON SUBLINEAR EXTENSIONS OF FUNCTIONALS

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1. A sequence $\{p_n\}_1^\infty$ of sublinear functionals defined on a vector subspace E_0 of a Fréchet space E* will be called **consistent** (with the topology of E) if, for every convergent sequence $\{x_n\}_1^\infty$ of elements of E_0, $\lim_{n\to\infty} x_n = x_0$, from the relations

$$\lim_{m,n\to\infty} p_k(x_n - x_m) = 0, \qquad k = 1, 2, \dots,$$

it follows that

$$x_0 \in E_0, \quad \lim_{n\to\infty} p_k(x_n - x_0) = \lim_{m\to\infty} p_k(x_0 - x_m) = 0, \qquad k = 1, 2, \dots .$$

We shall call a functional $\tilde{p}_k$ an **extension** of p_k from E_0 to E if $\tilde{p}_k$ is defined, sublinear, and continuous on E, and coincides with p_k on E_0. We shall call this extension **minimal** if there exists no extension q_k, different from $\tilde{p}_k$, for which $q_k(x) \le \tilde{p}_k(x)$, $x \in E$.

The result of this article is the following:

Theorem. *Let $\{p_k\}_1^\infty$ be a consistent sequence of sublinear functionals on a vector subspace E_0 of a Fréchet space E, and let E_0 be of the second category in itself. Then each of the functionals p_k admits a minimal extension $\tilde{p}_k$ from E_0 to E. Moreover, if $\psi_k(x)$ is a continuous seminorm on E such that $|p_k(x)| \le \psi_k(x)$ for all $x \in E_0$, then $\tilde{p}_k$ can be chosen so that $|\tilde{p}_k(x)| \le \psi_k$ for all $x \in E$.****

2. Let us note that this theorem contains three fundamental principles of linear analysis. Let us verify this.

2a. Let E and E_1 be Fréchet spaces; let $\{Q_n\}_1^\infty$ be a sequence of seminorms defining the topology in E_1; let $\{A_\alpha\}_{\alpha\in S}$ be a collection of linear continuous operators mapping E into E_1, and suppose, by assumption, that $\{A_\alpha x\}_{\alpha\in S}$ is a bounded set for each $x \in E$. The latter means that $p_n(x) = \sup_{\alpha \in S} Q_n(A_\alpha x) < \infty$ for any $x \in E$, $n = 1, 2, \ldots.$ It is verified directly that $\{p_n\}_1^\infty$ is a consistent sequence of sublinear functionals defined on $E_0 = E$. From our theorem it follows directly that the functionals p_n are continuous on E (and hence bounded on bounded subsets of E). Hence, $\lim_{x \to 0} A_\alpha x = 0$ uniformly in $\alpha \in S$.

Thus, the theorem contains the principle of uniform boundedness.

2b. Let E and E_1 be Fréchet spaces; let $\{Q_n\}_1^\infty$ be a sequence of seminorms defining the topology in E_1; let E_0 be a vector subspace dense in E, of the second category in itself; let $A : E_0 \to E_1$ be a closed linear operator ($AE_0 \subset E_1$). Put $p_n(x) = Q_n(Ax)$, $x \in E_0$, $n = 1, 2, \ldots.$ It is clear that $\{p_n\}_1^\infty$ is a compatible sequence of sublinear functionals defined on E_0. By our theorem all the functionals p_n are continuous on E_0. This means that the operator A is continuous on E_0 (from the fact that E_0 is dense in E, it also follows that $E = E_0$).

We see that this theorem contains the theorem on the closed graph.

2b. Let $p_0(x)$ be a discontinuous linear functional on a vector subspace E^0 of a Fréchet space E, and let E_0 be the closure of E^0 in E. Since p_0 admits an obvious discontinuous and linear extension to E_0, we may assume that the functional p_0 is defined on E_0. Putting in our theorem $p_k = p_0$, $k = 1, 2, \ldots$, we see that it implies the existence for p_0 of a minimal extension p from E_0 to E. Below it is verified that the minimality of the sublinear functional p implies its linearity. If, in addition, $|p_0(x)| \le \psi(x)$, $x \in E_0$, where ψ is a continuous seminorm on E, then, by our theorem, p can be chosen so that $|p(x)| \le \psi(x)$, $x \in E$. This gives us the Hahn–Banach theorem (in analytic form—for Fréchet spaces).

We shall verify the linearity of p by contradiction. Suppose that there exist $x_1, y_1 \in E$ such that $p(x_1 + y_1) < p(x_1) + p(y_1)$. The linear span $\widetilde{E}$ of the pair of elements x_1, y_1 is at most two-dimensional. It is immediately seen that in $\widetilde{E}$ there exists a linear functional f_1 for which $f_1(x) \le p(x)$ for all $x \in \widetilde{E}$. Form the functional $p_*(x) = \inf_{y \in \widetilde{E}}[f_1(y) + p(x - y)]$, $x \in E$. It is directly verified that the functional p_* is finite and sublinear on E, and $p_*(x) \le p(x)$ for $x \in E$, $p_*(y) \le f_1(y)$ for $y \in \widetilde{E}$. Since for $y \in \widetilde{E}$ we have $p_*(y) \le f_1(y) \le -p_*(-y) \le p_*(y)$, it follows that $p_*(y) = f_1(y)$ for $y \in \widetilde{E}$. For $z \in E_0$ we have $-p(-z) \le -p_*(-z) \le p_*(z) \le p(z)$ and $\pm p_0(z) = p_0(\pm z) = p(\pm z)$. Hence p_* is a sublinear extension from E_0 to E, and $p_*(x) \le p(x)$ for $x \in E$. Since $p_* = f_1$

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* A Fréchet space is a separable locally convex space E with a countable set of seminorms determining a metric in which E is complete.

** Note that the consistency property of the sequence of functionals $\{p_k\}_1^\infty$, upon extension to $\{\tilde{p}_k\}_1^\infty$, is preserved by virtue of the continuity of the latter.

on $\widetilde{E}$, we have

$$p_*(x_1) + p_*(y_1) = p_*(x_1 + y_1) \leq p(x_1 + y_1) < p(x_1) + p(y_1),$$

whence either $p_*(x_1) < p(x_1)$, or $p_*(y_1) < p(y_1)$.

The contradiction obtained contradicts the minimality of the extension p, which proves its linearity (cf. (1), pp. 1197—1198).

We have verified that the theorem formulated here contains the three basic principles of linear analysis. Its proof is given below. It relies only on Zorn' s theorem on ordered spaces and on Baire' s theorem on categories.

3. Proof of the theorem. a) Let $\{P_n\}_1^\infty$ be a sequence of seminorms defining the topology in E, and let $\{p_n\}_1^\infty$ be the sequence of functionals indicated in the theorem. Form the functionals

$$p_0(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|p_k(x)|}{1 + |p_k(x)|}, \quad x \in E_0; \qquad P(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{P_n(x)}{1 + P_n(x)}, \quad x \in E.$$

The functional P may be chosen as a quasinorm in E.* P defines a metric $r(x, y) = P(x - y)$, determining the topology of E, and, by assumption, E is complete in this metric. The functional p_0 has in E_0 all the properties of a quasinorm, except positivity on all elements of $E_0 \setminus \{0\}$. The compatibility of the sequence $\{p_n\}_1^\infty$ can be expressed by the requirement: if $\{x_n\}_1^\infty \subset E_0$, $\lim_{n\to\infty} x_n = x_0$, and $\lim_{n,m\to\infty} p_0(x_n - x_m) = 0$, then

$$x_0 \in E_0 \quad \text{and} \quad \lim_{n\to\infty} p_0(x_n - x_0) = 0.$$

Denote $P_0(x) = \max[p_0(x), P(x)]$, $x \in E_0$. P_0 may be chosen as a quasinorm on E_0; moreover $P_0(x) \geq P(x)$ for $x \in E_0$. From the

* A functional P is called a quasinorm in E if it is subadditive, $P(0) = 0$, $P(-x) = P(x)$, and $\lim_{\lambda\to 0} P(\lambda x) = 0$ for every $x \in E$; $\lim_{n\to\infty} P(x_n) = 0$ implies $\lim_{n\to\infty} P(\lambda x_n) = 0$ for $\{x_n\}_1^\infty \subset E$ and $-\infty < \lambda < \infty$, $P(x) > 0$ for $x \in E \setminus \{0\}$.

from the consistency of $\{p_n\}_1^\infty$ it follows that E_0 is complete in the metric $\rho(x, y) = P_0(x - y)$.

Thus, E_0 is considered by us in two topologies: in the P-topology it is of second category in itself, while in the P_0-topology it is complete; moreover, $P(x) \leq P_0(x)$ for $x \in E_0$. Applying under these conditions, for example, the arguments from (^{2}), pp. 64-65, where only Baire' s category theorem is used, we see that for every $\varepsilon > 0$ one can indicate a $\delta(\varepsilon) > 0$, $\delta(\varepsilon) \leq \varepsilon$, such that $P(x) \leq \delta(\varepsilon)$ entails $P_0(x) \leq \varepsilon$ $(x \in E_0)$. A fortiori,

$$P(x) \leq \delta(\varepsilon) \quad \text{entails} \quad p_0(x) \leq \varepsilon. \tag{1}$$

For a natural number k, consider $\varepsilon = 1/2^{k+1}$ and denote $\delta_k = \delta(\varepsilon)$. Choose a natural number N_k so that $1/2^{N_k} < \delta_k$, and denote

$$\eta_k = (\delta_k - 2^{-N_k})/(1 - \delta_k).$$

In addition, for what follows denote

$$\chi_k(x) = \frac{1}{\eta_k} \max_{1 \le m \le N_k} P_m(x) \quad (x \in E). \tag{2}$$

Let us note that from $\chi_k(x) \le 1,\ x \in E_0$, it follows that

$$\sum_{m=1}^{N_k} \frac{1}{2^m} \frac{P_m(x)}{1 + P_m(x)} \le \frac{\eta_k}{1 + \eta_k} \left(1 - \frac{1}{2^{N_k}}\right).$$

Since, moreover,

$$\sum_{m=N_k+1}^{\infty} \frac{1}{2^m} \frac{P_m(x)}{1 + P_m(x)} \le \frac{1}{2^{N_k}},$$

we have

$$P(x) \le \frac{\eta_k}{1 + \eta_k} \left(1 - \frac{1}{2^{N_k}}\right) + \frac{1}{2^{N_k}} = \frac{\eta_k + 1/2^{N_k}}{1 + \eta_k} = \delta_k = \delta(\varepsilon).$$

Therefore, from $\chi_k(x) \le 1,\ x \in E_0$, it follows that $p_0(x) \le \varepsilon$, and this, by virtue of the definition of $p_0(x)$, gives

$$|p_k(x)|/(1 + |p_k(x)|) \le 2^k \varepsilon.$$

Since we chose $\varepsilon = 1/2^{k+1}$, we obtain $|p_k(x)| \le 1$.

Thus, $\chi_k(x) \le 1,\ x \in E_0$, entails $|p_k(x)| \le 1$. From the positive homogeneity of the functionals χ_k and p_k it follows that

$$|p_k(x)| \le \chi_k(x) \quad \text{for } x \in E_0. \tag{3}$$

Formula (3) shows that p_k is continuous on $E_0,\ k = 1, 2, \ldots$.

b) Let $\psi_k(x)$ be some symmetric and continuous sublinear functional on E satisfying $|p_k(x)| \le \psi_k(x)$ for $x \in E_0$, for example, $\psi_k = \chi_k$. It is not hard to verify that the functional

$$q_k(x) = \inf_{y \in x - E_0} [p_k(x-y) + \psi_k(y)], \quad x \in E, \tag{4}$$

is sublinear (this is the enveloping sublinear functional for p_k and ψ_k)* . Moreover, $q_k(x) \leq \psi_k(x)$ for $x \in E$ and $q_k(y) \leq p_k(y)$ for $y \in E_0$; the first of these inequalities entails the continuity of q_k in E. In addition, for any $\varepsilon > 0$ one can indicate such a $y_\varepsilon \in x - E_0$ that, for the given $x \in E$,

$$p_k(x - y_\varepsilon) + \psi_k(y_\varepsilon) < q_k(x) + \varepsilon.$$

Hence, for $x \in E_0$,

$$p_k(x) \leq \psi_k(y_\varepsilon) - p_k(y_\varepsilon) + p_k(x) \leq p_k(x - y_\varepsilon) + \psi_k(y_\varepsilon) < q_k(x) + \varepsilon.$$

In view of the arbitrariness of $\varepsilon > 0$, for the given $x \in E_0$ we obtain $p_k(x) \leq q_k(x)$. The inequality in the other direction was noted above. Thus,

$$q_k(x) = p_k(x) \quad \text{for } x \in E_0.$$

Thus, we have proved the existence of a (continuous and sublinear) extension q_k of the functional p_k from E_0 to E. Moreover, it has been proved that for every symmetric and continuous sublinear functional $\psi_k(x)$ in E satisfying $|p_k(x)| \leq \psi_k(x),\ x \in E_0$, $q_k(x)$ can be chosen so that $q_k(x) \leq \psi_k(x)$ for $x \in E$; since $-q_k(x) \leq q_k(-x) \leq \psi_k(-x) = \psi_k(x)$, it follows that $|q_k(x)| \leq \psi_k(x)$ for $x \in E$.

c) Let K denote the set of all extensions $\tilde{p}_k$ of the functional p_k from E_0 to E for which $\tilde{p}_k(x) \leq q_k(x)$ for $x \in E$; obviously, $q_k \in K$.

Order K as follows: if $r_1, r_2 \in K$ and $r_1(x) \geq r_2(x)$ for all $x \in E$, then we shall write $r_1 \prec r_2$, where $\prec$ is the symbol of the order relation.

We shall prove that the ordered space K is inductive, i.e., every linearly ordered subset $\{r_t\}_{t<\vartheta}$ of the space K has a majorant in K. By Zorn' s theorem it will then follow that there exists a maximal element in K; this element will be the required minimal extension $\tilde{p}_k$ of the functional p_k from E_0 to E. Moreover, $\tilde{p}_k(x) \leq q_k(x) \leq \psi_k(x)$, and therefore $|\tilde{p}_k(x)| \leq \psi_k(x)$ for $x \in E$.

Let $t_1 \prec t_2 \prec \vartheta$. From the subadditivity of r_{t_1} and r_{t_2} it follows that $-r_{t_1}(-x) \leq -r_{t_2}(-x) \leq r_{t_2}(x) \leq r_{t_1}(x)$ for $x \in E$.

Hence it follows that the functional $r_\vartheta(x) = \inf_{t<\vartheta} r_t(x)$, $x \in E$, is finite, and also $r_\vartheta(x) \leq r_t(x)$ for $t < \vartheta$ and $x \in E$. If $x, y \in E$, then $r_\vartheta(x+y) \leq r_t(x+y) \leq r_t(x) + r_t(y)$ for $t < \vartheta$. Therefore the functional r_ϑ is finite and subadditive; its

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* See (^{1}), pp. 1199-1200, where verification of sublinearity is given.

positive homogeneity is obvious. Thus $r_\vartheta \in K$; moreover, r_ϑ is a majorant of $\{r_t\}_{t<\vartheta}$ in the sense of the order $\prec$.

This proves that p_k has a minimal extension $\tilde{p}_k$ from E_0 to E (cf. (1), p. 1198), and in the case $|p_k(x)| \leq \psi_k(x)$ for $x \in E_0$ one can choose $\tilde{p}_k$ so that $|\tilde{p}_k(x)| \leq \psi_k(x)$ for $x \in E$.

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Note: Figure translations are in progress. See original paper for figures.