# Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

## Abstract

## Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 187, No. 3

UDC 519.212.3

*MATHEMATICS*

R. V. Ambartsumian

# INTERSECTIONS OF COMPLEX CURVES BY RANDOM STRAIGHT LINES

*(Presented by Academician Yu. V. Linnik, 27 XII 1968)*

The classical Crofton theorem on the number of intersections of a piecewise-smooth plane curve $\mathcal{L}$ with straight lines states that

$$\int n(g)\,\dot{g} = 2L. \tag{1}$$

Here $\dot{g}$ is the (normalized) invariant density of lines on the plane with respect to motions; $n(g)$ is the number of intersections of the line $g$ with the curve $\mathcal{L}$, and $L$ is the length of the curve $\mathcal{L}$ (see ([1])).

Equality (1) is given the form of a probabilistic statement when only lines intersecting some convex region $D$ containing $\mathcal{L}$ entirely are considered. Indeed, introducing the function of lines

$$\delta_D(g) = \begin{cases} 1, & \text{if } g \text{ intersects } D, \\ 0, & \text{otherwise,} \end{cases}$$

on the basis of (1) one may write

$$\int n(g)\,\frac{\dot{g}\delta_D(g)}{U} = 2\frac{L}{U}. \tag{2}$$

Here and below $U$ is the length of the perimeter of $D$, and the integral on the left is understood as the mean number of intersections of $\mathcal{L}$ with a random line having probability distribution with density $\dot{g}\delta_D(g)U^{-1}$. In what follows, a random line with such a distribution will be denoted by $\mathcal{M}$.

Although results (1) and (2) are 100 years old, little has been known up to now about the distribution of the number of intersections of the random line $\mathcal{M}$ with

$\mathcal{L}$. Almost the only results in this direction are those of Sylvester ([2]) for the case when $\mathcal{L}$ is a union of three nonintersecting convex contours, i.e., when a direct reduction of the problem to Crofton's result on mean numbers is possible, and of Zuluaga ([6]), who considered the relation between the topological properties of curves and the supports of the distribution of the number of their intersections with $\mathcal{M}$.

The method of invariant imbedding, first applied in problems of geometry in ([3,4]), leads to the determination of the distribution of the number of intersections of $\mathcal{L}$ with $\mathcal{M}$ under the most general assumptions concerning $\mathcal{L}$. In saying this, we have in mind the reduction of integration over a two-dimensional manifold to integration over one- and zero-dimensional manifolds. In particular, equalities (1) and (2) have the same meaning.

The method of invariant imbedding itself is directly applicable only when $\mathcal{L}$ is taken to be a polygonal line, and consists in the fact that the problem for the random line $\mathcal{M}$ is considered as a limit problem for an analogous problem posed for a suitable random circle, when the radius of the circle tends to $\infty$.

Let us make this statement precise. By definition, a random circle $C(r)$ has constant radius $r$, and its center is distributed uniformly inside the "fundamental disk" of radius $R$ with center at the origin. $R$ is chosen so large that the distance from $\mathcal{L}$ to the boundary of the fundamental disk exceeds $r$.

Let $p_k(r)$ denote the probability of $k$ intersections of $C(r)$ with $\mathcal{L}$; $p_D(r)$, the probability of intersection of $C(r)$ with the convex domain $D$; $p_k$, the probability of $k$ intersections of $\mathcal{M}$ with $\mathcal{L}$.

**Lemma.** Suppose that $\mathcal{L}$ is a broken line such that no three of its vertices lie on one straight line, and $D$ is a convex polygon containing $\mathcal{L}$. Then the limit of the ratio $p_k(r)/p_D(r)$ exists (this ratio does not depend on $R$) as $r \to \infty$, $k > 0$, and

$$p_k = \lim_{r\to\infty} \frac{p_k(r)}{p_D(r)} \quad \text{for } k > 0.$$

The derivatives $\dfrac{d}{dr}p_k(r)$ and $\dfrac{d}{dr}p_D(r)$ exist, and

$$p_k = \lim_{r\to\infty} \frac{\dfrac{d}{dr}p_k(r)}{\dfrac{d}{dr}p_D(r)} \quad \text{for } k > 0. \tag{3}$$

Using representation (3) to compute $p_k$ for broken lines leads to the goal without special difficulties. In this, finding the asymptotics of the derivatives appearing in (3) is based on a number of simple geometric considerations, such as, for example, the fact that more than one vertex of the broken line $\mathcal{L}$ falls into the annulus bounded by the circle $C(r+h)$ and the concentric circle of radius $r$ with probability of order of smallness $o(h)$.

After computing the values $p_k$ for broken lines, it is easy to carry out formally, and then justify, the limiting transition to piecewise-smooth curves. The result has an especially simple form in the case when $\mathcal{L}$ is a closed, sufficiently smooth curve of finite length.

We additionally require that:

a) on $\mathcal{L}$ there be no three distinct points at which the tangents to $\mathcal{L}$ coincide (absence of triple tangents);

b) the curve $\mathcal{L}$ have only a finite number of inflection points;

c) the curve $\mathcal{L}$ have only a finite number of straight lines tangent to $\mathcal{L}$ at two distinct points (double tangents);

d) the number of intersections of $\mathcal{L}$ with a straight line be bounded.

It is not excluded that the curve $\mathcal{L}$ consists of a finite number of closed components (as curves). We formulate the result for such curves. To this end we introduce three new random straight lines: $T$, $W^+$, $W^-$. The random straight line $T$, by definition, is a random tangent to $\mathcal{L}$, and the point of tangency of $T$ with $\mathcal{L}$ is distributed uniformly along $\mathcal{L}$.

By $R_i$ (respectively, $Q_i$) we denote all those double tangents for which the curve $\mathcal{L}$, locally, near the points of tangency, lies on one side (respectively, on different sides) of the double tangent itself. By $r_i$ (respectively, $q_i$) we denote the length of the segment between the points of tangency of $R_i$ (respectively, $Q_i$) with $\mathcal{L}$.

The random straight line $W^+$, by definition, takes the position $R_i$ with probability equal to $r_i/r$, $r=\sum r_i$.

The random straight line $W^-$, by definition, takes the position $Q_i$ with probability equal to $q_i/q$, $q=\sum q_i$.

Let $t_k$ denote the probability of $k$ intersections of the random straight line $T$ with $\mathcal{L}$ (not counting the point of tangency); $w_k^+$, the probability of $k$ intersections of the random straight line $W^+$ with $\mathcal{L}$ (not counting the points of tangency); $w_k^-$, the probability of $k$ intersections of the random straight line $W^-$ with $\mathcal{L}$ (not counting the points of tangency).

**Theorem*.** For every smooth closed curve $\mathcal{L}$ of finite length satisfying conditions a), b), c), and d), the relation

$$p_k = \frac{r}{U}\left[2w_{k-2}^+ - w_k^+ - w_{k-4}^+\right] - \frac{q}{U}\left[2w_{k-2}^- - w_k^- - w_{k-4}^-\right] +$$

* The corresponding results for more general (piecewise-smooth) curves can be obtained by approximating them by curves satisfying the conditions of the theorem.

$$+\frac{L}{U}[t_{k-2} - t_k], \qquad k > 0; \tag{4}$$

$$p_0 = 1 - \frac{r}{U}w_0^+ + \frac{q}{U}w_0^- - \frac{L}{U}t_0.$$

Probabilities with negative indices should be taken to be equal to zero.

Let us note that a closed curve intersects any line in an even number of points (see [6]). If $2N$ is the maximal number of intersections that can occur when $\mathcal{L}$ is intersected by a line (i.e., $p_{2N} > 0$, but $p_{2n} = 0$ if $n > N$), then it is obvious that $t_{2n} = 0$ for $n > N-1$ and $w_{2n}^+ = w_{2n}^- = 0$ for $n > N-2$. Taking this into account, from (4) we easily find the mathematical expectation of the random number $\xi$ of intersections of $\mathcal{M}$ with $\mathcal{L}$:

$$\bar{n} = E\xi = \sum kp_k = 2L/U,$$

which coincides with (2). Further, putting

$$\bar{m} = \sum kt_k,$$

we find the second moment of the distribution $p_k$:

$$E\xi^2 = \frac{8(q-r)}{U} + 4\frac{L}{U}(\bar{m}+1).$$

Thus the variance $D\xi$, important for computational applications, is equal to

$$D\xi = \frac{8(q-r)}{U} + 4\frac{L}{U}(\bar{m}+1) - 4\frac{L^2}{U^2},$$

and the coefficient of variation, consequently, has the form

$$D\frac{\xi}{\bar{n}} = \frac{2(q-r)U}{L^2} + \frac{U}{L}(\bar{m}+1) - 1. \tag{5}$$

When estimating the magnitude of the coefficient of variation for a given $\mathcal{L}$, it is important in practice to be able to neglect the first term in its expression (5). We confine ourselves to the remark that if $\mathcal{L}$ is the union of some number of circles, then $q - r < 0$, i.e., the estimate

$$D\frac{\xi}{\bar{n}} < \frac{U}{L}(\bar{m}+1) - 1$$

holds.

It is of interest to indicate a broader class of curves for which this estimate remains valid.

For the complete proofs, omitted in the present note, see [5].

Institute of Mathematics and Mechanics
Academy of Sciences of the ArmSSR

Received
20 XII 1968

## CITED LITERATURE

[1] W. Blaschke, *Vorlesungen über Integralgeometrie*, N. Y., 1949. [2] J. J. Sylvester, Acta Math., **14**, 185 (1891). [3] R. V. Ambartsumian, *Probability Distributions in Geometric Combinatorics*, Preprint, Institute of Mathematics and Mechanics, Academy of Sciences of the ArmSSR, Yerevan, 1968. [4] R. V. Ambartzumian, Studia Sci. Math. Hung., in press. [5] R. V. Ambartsumian, *Intersections of a Plane Curve by Random Secants and Tangents*, Preprint, VINITI, 1968. [6] R. Sulanke, Acta Math. Acad. Sci. Hung., **17** (3–4), 233 (1966).

*Note: Figure translations are in progress. See original paper for figures.*