G. D. Karatopragliev

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Abstract

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G. D. Karatoprakliev

ON A BOUNDARY-VALUE PROBLEM FOR EQUATIONS OF MIXED TYPE IN MULTIDIMENSIONAL DOMAINS

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Boundary-value problems for equations of mixed type in multidimensional domains have been considered in works by a number of authors ($^{1-6}$)*. In the present note the existence of weak solutions and the uniqueness of a smooth solution of one boundary-value problem for equations of mixed type in a bounded multidimensional domain are proved.

Let G be a bounded domain of the n-dimensional space E_n with piecewise smooth boundary Γ, divided by the plane $x_n = 0$ into two domains $G_1 = G \cap \{x_n > 0\}$ and $G_2 = G \cap \{x_n < 0\}$, with $\Gamma \cap S_0 = 0$, where $S_0 = G \cap \{x_n = 0\}$. Denote by Σ and S those parts of Γ which lie respectively in the half-spaces $x_n \geq 0$ and $x_n < 0$.

Consider the operator

$$Lu = \begin{cases} a^{ij}(x)u_{x_i x_j} + b^i(x)u_{x_i} + c(x)u, & \text{for } x \in \overline{G}_1, \\ k(x_n)\Delta u + u_{x_n x_n}, & \text{for } x \in \overline{G}_2, \end{cases}$$

where Δ is the Laplace operator in E_{n-1} (summation over repeated indices is assumed everywhere from 1 to n). We shall assume that $a^{ij}(x) \in C^2(\overline{G}_1)$, $a^{ij} = a^{ji}$; $b^i(x) \in C^1(\overline{G}_1)$; $c(x) \in C(\overline{G}_1)$; $k(x_n)$ is continuous on the interval $[-h, 0]$, $-h = \inf_{x \in G_2} x_n$, and continuously differentiable in $[-h, 0)$, with $k(x_n) < 0$ and $k'(x_n) > 0$ in $[-h, 0)$,

$$\lim_{x_n \to 0} \frac{k(x_n)}{k'(x_n)} = 0,$$

$[k(x_n)/k'(x_n)]'$ is summable and bounded on the interval $[-h, 0]$; the coefficients of the operator L are continuous in passing through the plane $x_n = 0$; $a^{nn}_{x_n} = 0$ on S_0; $a^{ij}(x)\xi_i\xi_j \geq 0$ in $\overline{G}_1$ for all $(\xi_1, \dots, \xi_n)$. Thus, in $\overline{G}_1$ the operator L is elliptic-parabolic, while in $G_2 \cup S$ it is hyperbolic with parabolic degeneration on S_0.

Let $\nu = (\nu_1, \dots, \nu_n)$ be the unit vector of the inner normal to Γ. Introduce the following notation: Σ^0 is the set of those points of Σ at which $a^{ij}\nu_i\nu_j = 0$; $b(x) = (b^i - a^i_{x_j})\nu_i$ for $x \in \Sigma^0$; Σ_0 is the set of those points of Σ^0 where $b = 0$; $\Sigma^1 = \Sigma^0 \setminus \Sigma_0$; Σ_1 is the set of those points of Σ^1 where $b > 0$; $\Sigma_2 = \Sigma^1 \setminus \Sigma_1$; $\Sigma_3 = \Sigma \setminus \Sigma^0$.

Denote by L^+ the operator adjoint to L:

$$L^+u = \begin{cases} a^{ij}(x)u_{x_ix_j} + b^{+i}(x)u_{x_i} + c^+(x)u, & \text{for } x \in \overline{G}_1, \\ k(x_n)\Delta u + u_{x_nx_n}, & \text{for } x \in \overline{G}_2, \end{cases}$$

where

$$b^{+i} = 2a^{ij}_{x_j} - b^i, \qquad c^+ = a^{ij}_{x_ix_j} - b^i_{x_i} + c.$$

Let $W_2^2(\text{gr})$ be the set of functions $u \in W_2^2(G)$ satisfying the condition $u = 0$ on $\Sigma_2 \cup \Sigma_3 \cup S$, and let $W_2^2(\text{gr})^+$ be the set of those $v \in W_2^2(G)$,

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* In work ([4]) the question of uniqueness of the solution of one boundary-value problem for the equation $k(z)(u_{xx}+u_{yy})+u_{zz}=0$ is considered. Unfortunately, there is an error in the proof (the author' s assertion that on the surface S_4 the integral I_4 is positive definite is clearly incorrect).

for which $(Lu, v)_0 = (u, L^+v)_0$ for every $u \in W_2^2(\text{gr})$ (by $(\ ,\ )_0$ we denote the scalar product in $L_2(G)$). If S contains no pieces of characteristics, it is easy to see that the functions $v \in W_2^2(\text{gr})^+$ satisfy the condition $v = 0$ on $\Sigma_1 \cup \Sigma_3 \cup S$. Let $f(x) \in L_2(G)$.

We shall call a function $u \in L_2(G)$ a weak solution of the problem

$$Lu = f \text{ in } G, \qquad u = 0 \text{ on } \Sigma_2 \cup \Sigma_3 \cup S, \tag{1}$$

if $(u, L^+v)_0 = (f, v)_0$ for every $v \in W_2^2(\text{gr})^+$.

It is known (see (7)) that, for the existence of a weak solution of problem (1) for every $f \in L_2$, it is necessary and sufficient that the inequality

$$\|L^+v\|_0 \geq C\|v\|_0, \qquad v \in W_2^2(\text{gr})^+, \qquad C > 0$$

hold. If the stronger inequality

$$\|L^+v\|_0 \geq C\|v\|_+, \qquad v \in W_2^2(\text{gr})^+, \qquad C > 0, \tag{2}$$

holds, where $\|v\|_+$ is a positive norm in some Hilbert space H_+, then there exists a weak solution for every $f \in H_-$, where H_- is the space with negative norm

$$\|f\|_- = \sup_{v \in H_+} [|(f, v)_0|/\|v\|_+]$$

(the terminology adopted in (7) is used in this paper).

We shall show that there exist such domains G_2 with piecewise smooth boundary S, containing no pieces of characteristics, that in G inequality (2) will hold.

Let $v \in W_2^2(\mathrm{gr})^+$; $p(x_n) = -(x_n + d)^\theta$ in $\overline{G}$, where d and θ are constants, with $d \geq d_0 > -h$ sufficiently large, and $0 < \theta < 1$; $q(x_n) = -pr$ in $[-h, 0]$, where $r = -4k/k'$. Integrating by parts and taking into account that $a^{ij} v_j = 0$ on Σ^0 and $v_{x_i} = N_v(x) v_i$, for $x \in S$, we obtain

$$\int_{G_1} pvL^+v\,dx = \int_{G_1} \left[p\left(c + \frac{1}{2}a^{ij}_{x_i x_j} - \frac{1}{2}b^i_{x_i}\right) + \frac{1}{2}p'b^n + \frac{1}{2}p''a^{nn}\right] v^2\,dx - \int_{G_1} pa^{ij}v_{x_i}v_{x_j}\,dx + \int_{\Sigma_2} pbv^2\,ds - p(0)\int_{S_0} vv_{x_n}\,ds + \frac{p'(0)}{2}\int_{S_0} v^2\,ds, \tag{3}$$

$$\int_{G_2} (pv + qv_{x_n})L^+v\,dx = \frac{1}{2}\int_{G_2} p''v^2\,dx + \frac{1}{2}\int_{G_2} [(pr)' - 2p]\left(-k\sum_{i=1}^{n-1} v^2_{x_i} + v^2_{x_n}\right) dx + \frac{1}{2}\int_S N_v^2 prv_n\left(k\sum_{i=1}^{n-1} v_i^2 + v_n^2\right) ds + p(0)\int_{S_0} vv_{x_n}\,ds - \frac{p'(0)}{2}\int_{S_0} v^2\,ds. \tag{4}$$

Adding (3) and (4), we obtain

$$\int_G p(v - \chi rv_{x_n})L^+v\,dx = \int_{G_1} \varphi v^2\,dx + \frac{1}{2}\int_{G_2} p''v^2\,dx - \int_{G_1} pa^{ij}v_{x_i}v_{x_j}\,dx + \frac{1}{2}\int_{G_2} [(pr)' - 2p]\left(-k\sum_{i=1}^{n-1} v_{x_i} + v_{x_n}\right) dx + I, \tag{5}$$

where

$$2\varphi(x) = p\left(2c + a^{ij}_{x_i x_j} - b^i_{x_i}\right) + p'b^n + p''a^{nn},$$

χ is the characteristic function of the domain G_2,

$$I = \int_{\Sigma_2} pbv^2\,ds + \frac{1}{2}\int_S N_v^2 prv_n\left(k\sum_{i=1}^{n-1} v_i^2 + v_n^2\right) ds = I_1 + I_2.$$

Obviously, $I_1 \geq 0$. The surfaces

$$\sum_{i=1}^{n-1} \alpha_i x_i + \int_0^{x_n} \sqrt{-k(t)}\, dt + \beta = 0,$$

$-h \le x_n < 0$, where a_i, β are arbitrary real numbers, and

$$\sum_{i=1}^{n-1} \alpha_i^2 = 1,$$

are characteristics of the operator L. Let T be the section of the cylinder $Q\{x \in S_0,\ x_n < 0\}$ by some characteristic surface, with $T \cap S_0 = 0$. Denote by M the set of those points of ∂T for which $x_n = x_n^0 = \sup_{x \in \partial T} x_n$ (∂T is the boundary of T). Draw, at some point $x^0 \in M$, a tangent plane to the characteristic surface. Taking into account the properties of the function $k(x_n)$, it is easy to see that the section S_2 of the cylinder Q by the tangent plane lies below the section T. Denote by S_1 the part of the boundary of the cylinder Q lying between S_2 and the plane $x_n = 0$. Let $S = S_1 \cup S_2$. Then $I_2 \ge 0$, since $\nu_n = 0$ on S_1, while on S_2 we have

$$\nu_n = \{k(x_n^0)/[k(x_n^0) - 1]\}^{1/2} > 0 \quad \text{and} \quad k(x_n) \sum_{i=1}^{n-1} \nu_i^2 + \nu_n^2 = k(x_n)(1 - \nu_n^2) + \nu_n^2$$

$$= [k(x_n^0) - k(x_n)]/[k(x_n^0) - 1] < 0.$$

Let us note that every section of the cylinder Q by a plane passing through the point x^0 and lying below S_2 can be taken as S_2. Indeed, on this section $0 < \nu_n < \{k(x_n^0)/[k(x_n^0) - 1]\}^{1/2}$, whence it follows that

$$k(x_n) \sum_{i=1}^{n-1} \nu_i^2 + \nu_n^2 < k(x_n) + [1 - k(x_n)]k(x_n^0)/[k(x_n^0) - 1] =$$

$$= [k(x_n^0) - k(x_n)]/[k(x_n^0) - 1] < 0.$$

Suppose that $\varphi(x) \ge \varepsilon = \text{const} > 0$ in $\overline{G}_1$. This condition is satisfied if, for example, the function $2c + a^{ij}_{x_i x_j} - b^i_{x_i}$ is negative and sufficiently large in absolute value. Let the Frankl condition be satisfied:

$$F(x_n) = 1 + 2(k/k')' \ge \delta = \text{const} > 0$$

for $-h \le x_n < 0$. Then

$$(pr)' - 2p \geq p'r - 2p\delta = \frac{-\theta r + 2(x_n + d)\delta}{(x_n + d)^{1-\theta}} \geq \frac{\delta'}{d^{1-\theta}} = \delta_1,$$

if $d \geq -x_n + (\delta' + \theta r)/2\delta$ for $-h \leq x_n < 0$, where $\delta' = \text{const} > 0$.

Taking into account that $p(x_n) \leq -\delta_2$ and $p''(x_n) \geq \delta_3$ in $\overline{G}$, where δ_2, δ_3 are positive constants, from (5) we obtain

$$C_2 \|L^+ v\|_0 \|v\|_+ \geq \int_G p(v - xrv_{x_n}) L^+ v \, dx \geq C_1 \|v\|_+^2,$$

where $C_i > 0$ $(i = 1, 2)$ and

$$\|v\|_+^2 = \int_G v^2 \, dx + \int_{G_1} a^{ij} v_{x_i} v_{x_j} \, dx + \int_{G_2} \left(-k \sum_{i=1}^{n-1} v_{x_i}^2 + v_{x_n}^2 \right) dx.$$

Hence (2) follows.

Thus the following holds.

Theorem 1. *There exist bounded domains G in E_n with a piecewise smooth boundary such that, if $\varphi(x) \geq \varepsilon > 0$ in $\overline{G}_1$ and $F(x_n) \geq \delta > 0$ in $[-h, 0)$, then for problem (1) inequality (2) holds. Consequently, there exists a weak solution of this problem for any $f \in H_-$.*

In an analogous way one derives the inequality

$$\|Lu\|_0 \geq C \|u\|_+, \qquad u \in W_2^2 \text{ (gr)}, \quad C > 0, \tag{6}$$

only the condition $\varphi(x) \geq \varepsilon$ must be replaced by the condition $\psi(x) \geq \varepsilon$, where

$$2\psi(x) = p(2c + a^{ij}_{x_i x_j} - b^i_{x_i}) + p'(2a^{nj}_{x_j} - b^n) + p'' a^{nn}.$$

A smooth solution of problem (1) is a function $u \in W_2^2(\text{gr})$ satisfying the equation $Lu = f$ almost everywhere in G. From inequality (6) it follows

Theorem 2. *There exist bounded domains G in E_n with piecewise smooth boundary such that, if $\psi(x) \geq \varepsilon > 0$ in $\overline{G}_1$ and $F(x_n) \geq \delta > 0$ in $[-h, 0)$, then problem (1) can have no more than one smooth solution.*

Remark. The results obtained also hold in the case when on S_0 only the coefficients at $u_{x_i x_n}$, $i = 1, \dots, n$, are continuous, and $b^n - a^{nn}_{x_n} = 0$ on S_0. The remaining coefficients of the operator L may have a discontinuity of the first kind when passing through S_0.

Mathematical Institute
with Computing Center
Sofia, Bulgaria

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Note: Figure translations are in progress. See original paper for figures.