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Abstract

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MATHEMATICS

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ON THE ROTATION OF MULTIVALUED VECTOR FIELDS

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1. A mapping F acting in a Banach space X is called multivalued if to each point $x \in X$ there corresponds a subset $F(x) \subset X$. A point $x_0 \in X$ is called a fixed point of the multivalued mapping F if $x_0 \in F(x_0)$. The problem of the existence of fixed points for multivalued mappings is of interest for various branches of mathematics (dynamical systems without uniqueness, game theory, etc.). A number of works are devoted to it ⁽¹⁻⁵⁾.

In the present article, for multivalued mappings with convex images, we introduce a new topological invariant—the notion of rotation of the corresponding vector field, closely connected with fixed-point theory. A number of properties analogous to the properties of the rotation of single-valued vector fields are established (on these questions see, for example, ⁽⁴⁾); applications to fixed-point theorems are given.

2. We first consider the finite-dimensional space $X = R_n$. Let \bar{G} be the closure of an open set G with triangulable boundary Γ . Suppose that a mapping F with convex images is defined on Γ and has the following properties: 1) it is closed (i.e., the graph

$$\bigcup_{x \in \Gamma} (x, F(x))$$

is closed in $X \times X$); 2) it is compact (i.e.,

$$\overline{\bigcup_{x \in \Gamma} F(x)}$$

is compact). Such mappings will be called completely continuous mappings, bearing in mind that this definition is to be extended below also to mappings in Banach spaces.

A multivalued mapping defines a multivalued vector field on the boundary Γ by the formula $\Phi(x) = x - F(x)$ (in the sense of the algebraic difference of two sets), which we shall also call completely continuous. If $x_0 \in F(x_0)$, then the

image $\Phi(x_0)$ contains Θ . Such a point x_0 will be called a special point of the multivalued vector field.

Lemma 1. *Let F be a completely continuous multivalued mapping $\Gamma \rightarrow R_n$, and let $x_j \rightarrow x$, $x_j \in \Gamma$. Then for every $\varepsilon > 0$ there is an n_0 such that $F(x_j) \subset F^\varepsilon(x)$, $j \geq n_0$, where $F^\varepsilon(x)$ is the ε -neighborhood of $F(x)$.*

We make a simplicial subdivision of the boundary Γ and choose an arbitrary simplex S_p of this subdivision. Consider the F -images $F(y_p^1), \dots, F(y_p^k)$ of its vertices. We fix arbitrarily points $z^r \in F(y_p^r)$ ($r = 1, \dots, k$) and span over them the convex hull Δ . Define a continuous single-valued mapping $f : S_p \rightarrow \Delta$ by the equality

$$f \left(\sum_{r=1}^k \lambda_r y_p^r \right) = \sum_{r=1}^k \lambda_r z^r \quad \left(\lambda_r \geq 0, \sum_{r=1}^k \lambda_r = 1 \right).$$

Thus we have constructed a continuous single-valued approximation $f : \Gamma \rightarrow R_n$ of the multivalued mapping F .

Lemma 2. *Under the condition that Φx has no special points on Γ , and for sufficiently fine simplicial subdivisions of the boundary Γ , the vector field $\varphi(x) = x - f(x)$ also has no special points on Γ .*

Proof. Let $\varepsilon > 0$ be the smallest distance between the points x and their F -images, existing by Lemma 1. Let $V(\Theta) =$

neighborhood of zero of radius $\eta < \varepsilon/2$. Then $V(x) \cap V(F(x)) = \emptyset$ for every $x \in \Gamma$. By $U(x)$ we denote the largest neighborhood of the point x such that $F(y) \subset V(F(x))$, if $y \in U(x)$ (its existence follows from Lemma 1). Let δ be a Lebesgue number of the cover $\{U(x) \cap V(x)\}$ of the boundary Γ ; choose a simplicial subdivision of Γ with the diameter of the simplexes less than $\delta/2$. Then each simplex S_p belongs to some neighborhood $U(x) \cap V(x)$, and the F -images of its vertices lie in $V(F(x))$. Consequently, the f -image of the simplex S_p lies in $V(F(x))$, whence the assertion of the lemma follows.

Below it is assumed that the space R_n is oriented. We formulate the basic

Definition 1. The **rotation** $\gamma(\Phi, \Gamma)$ of a multivalued field Φ on the boundary Γ will mean $\gamma(\varphi, \Gamma)$ of a single-valued vector field φ , where φ is constructed on a sufficiently fine simplicial subdivision of Γ .

It can be shown that the definition does not depend on the choice of the simplicial subdivision, the points z^r , and the radius of the separating neighborhood V .

Let us note that another approach is also possible, based on the notion of the degree of a multivalued mapping of the boundary Γ into the sphere S^{n-1} . First define the degree of such a mapping Ψ , which is assumed to be closed and for which the spherical convex hull of each image $\Psi(x)$ does not contain diametrically opposite points. In this case one can construct a single-valued

approximation of the mapping Ψ . To this end, for each point $x \in \Gamma$ we fix a neighborhood $U(x)$ so that the Ψ -images of the points $y \in U(x)$ belong to the ε_x -inflation of the image $\Psi(x)$, where ε_x is so small that the ε_x -inflation contains no diametrically opposite points. The cover $\{U(x)\}$ determines a Lebesgue number δ , and, if the triangulation is sufficiently fine, then a single-valued approximation ψ is constructed, as in the first case, affinely on each simplex.

Definition 2. The degree of the mapping $\psi : \Gamma \rightarrow S^{n-1}$ under a sufficiently fine triangulation of Γ will be called the **degree of the multivalued mapping** $\Psi : \Gamma \rightarrow S^{n-1}$.

It is not difficult to show that this definition is correct.

Definition 3. The **rotation of a multivalued vector field** Φ is called the degree of the multivalued mapping

$$\Psi(x) = \left\{ y : y = \frac{z}{\|z\|}, z \in \Phi(x) \right\}$$

of the boundary Γ into S^{n-1} .

Definitions 1 and 3 are equivalent.

3. A number of properties known for the rotation of a single-valued field ⁽⁴⁾ carry over to the notion of the rotation of a multivalued vector field.

Multivalued vector fields Φ_0 and Φ_1 will be called homotopic on Γ if on $\Gamma \times I$, where $I = [0, 1]$, there exists a completely continuous multivalued vector field $\Phi(x, t)$ such that $\Theta \in \Phi(x, t)$, $x \in \Gamma$, $t \in I$, $\Phi(x, 0) = \Phi_0(x)$, $\Phi(x, 1) = \Phi_1(x)$.

Theorem 1. *If Φ_0 is homotopic to Φ_1 on Γ , then $\gamma(\Phi_0, \Gamma) = \gamma(\Phi_1, \Gamma)$.*

The assertion of the theorem follows from the fact that in a neighborhood of each point $t_0 \in I$ one can construct a common single-valued approximation for $\Phi(x, t)$.

The following proposition may be regarded as a general criterion for the existence of a fixed point.

Theorem 2. *Let a multivalued completely continuous mapping F be given on the closed domain \overline{G} , with no fixed points on the boundary Γ , and let $\gamma(\Phi, \Gamma) \neq 0$, where Φ is the corresponding vector field. Then there exists a fixed point $x_0 \in G$ of the mapping F .*

From this theorem one easily obtains a generalization of S. Kakutani's theorem ⁽²⁾.

Theorem 3. If a multivalued completely continuous mapping F is given on the closed ball $T \subset R_n$ and, for every x belonging to the boundary S^{n-1} of the ball T , the condition $F(x) \cap T = \emptyset$ is satisfied, then there exists a fixed point $x_0 \in T$ of the mapping F .

Indeed, in this case the rotation $\gamma(\Phi, \Gamma)$ of the field Φ is equal to one.

Multivalued mappings, besides isolated fixed points, may have connected sets of fixed points. Both will be called fixed elements. A fixed element x_* will be called isolated if some neighborhood of it does not intersect other fixed elements. The rotation of a multivalued vector field on the boundaries of such neighborhoods is constant; we shall call it the index $\gamma(x_*)$ of the fixed element.

Theorem 4. If, on the closed domain \overline{G} , a multivalued completely continuous vector field Φ has a finite number of isolated fixed elements x_1, \dots, x_k and has no special points on the boundary Γ , then

$$\gamma[\Phi, \Gamma] = \sum_{i=1}^k \gamma(x_k).$$

4. In conclusion we consider completely continuous multivalued vector fields Φ in a Banach space (we retain the definition of Sec. 2). In this case Lemmas 1 and 2 remain valid. We construct a multivalued finite-dimensional approximation Φ_m of the field Φ .

Let V be the closed neighborhood of zero of radius $\rho < \varepsilon/2$. Choose in the range $F(\Gamma)$ a ρ -net $\{y_1, \dots, y_m\}$, and denote by K_m the convex hull, and by E_m the linear hull, of the net $\{y_1, \dots, y_m\}$. We shall assume that $G \cap E_m$ has a polyhedral boundary Γ_m for each E_m . Consider the multivalued vector field $\Phi_m(x) = (\Phi(x) + V) \cap K_m$ on Γ_m . This field, as can be shown, has no special points and satisfies the conditions of Sec. 2.

Theorem 5. The rotation $\gamma[\Phi_m, \Gamma_m]$ does not depend on the choice of the approximating space E_m or on the choice of $\rho < \varepsilon/2$.

It is therefore natural to give the following definition:

$$\gamma\{\Phi, \Gamma\} = \gamma\{\Phi_m, \Gamma_m\}.$$

The theorems proved in Sec. 3 are also valid in the case of a Banach space, in the very same formulations. We give the formulation of a theorem generalizing I. Glicksberg's theorem¹.

Theorem 6. If a multivalued completely continuous field $\Phi(x)$ is defined on the closed ball T of a Banach space, has no special points on the boundary Γ , and satisfies the condition $F(x) \cap T = \emptyset$, $x \in \Gamma$, then $\gamma[\Phi, \Gamma] = 1$; consequently, there exists a fixed point.

Applying the general schemes known in the theory of rotations of single-valued vector fields, one can also prove more general theorems on the existence of fixed points.

Remark. The condition of complete continuity of the mapping F in the preceding constructions can be somewhat weakened: the assumption of compactness of

the image $F(\Gamma)$ may be replaced by the condition of the existence of a compact set K intersecting each image $F(x)$, where $x \in \bar{G}$.

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