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Abstract

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MATHEMATICS

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ON PERFECT IRREDUCIBLE MAPPINGS OF COMPLETELY REGULAR SPACES

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All mappings are assumed to be continuous.

Definition 1. Let there be an inverse system $\Sigma = \{X_\alpha, \pi_{\alpha_2}^{\alpha_1}\}$ ($\alpha \in A$) of topological spaces X_α with continuous mappings $\pi_{\alpha_2}^{\alpha_1}$, constructed on the directed set $A = \{\alpha\}$. Suppose that in this set there is a minimal element α_0 , for which $\alpha \geq \alpha_0$ for every α . Then for every space X_α ($\alpha \in A$) there exists a mapping onto X_{α_0} . The limit of the system X can also be naturally mapped onto X_{α_0} . We shall say that the mapping $\pi : X \rightarrow X_{\alpha_0}$ is **decomposed into a system of mappings** $\pi_{\alpha_0}^\alpha : X_\alpha \rightarrow X_{\alpha_0}$.

Definition 2. A mapping $f : X \rightarrow Y$ will be called **elementary** if in X there is an open everywhere dense set U on which f is a homeomorphism.

§ 1. The extension d_{XY} and its properties

Let X and Y be locally bicomact spaces; $f : X \rightarrow Y$ a perfect irreducible mapping; bX a bicomact extension of the space X ; α a proximity space on X generated by the extension bX . Define on Y a proximity space $\beta : A\bar{\beta}B$ if and only if

$$(f^{-1}A)\alpha(f^{-1} - B).$$

All the axioms of a proximity space are easily verified.

The proximity space β corresponds to a bicomact extension of the space Y , which we shall denote by $d_{bX}Y$, or, when it is clear which bX and f are meant, simply by dY .

We establish some properties of the extension.

Theorem 1. *If the mapping $f : X \rightarrow bY$ can be extended to a mapping $g : bX \rightarrow bY$, then there exists a mapping $h : d_{bX}Y \rightarrow bY$, identical on Y .*

The proof follows easily from the definition of $d_{bX}Y$.

Theorem 2. Let g be the embedding $Y \rightarrow d_{bX}Y$; then the mapping $fg : X \rightarrow dY$ can be extended to a mapping $bX \rightarrow dY$.

Part of the proof of Theorem 2. Let $x \in bX$, and let δ be the system of all neighborhoods of the point x : $\delta = \{S_\alpha\}_{\alpha \in A}$. Put $T_\alpha = f^\#(S_\alpha \cap X)$ —this is a nonempty open set by virtue of the irreducibility and closedness of the mapping f . If there exists a finite family of sets $T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, \dots, T_{\alpha_n}$, then

$$f^{-1} \left(\bigcap_{k=1}^n T_{\alpha_k} \right) \subset X \cap \left(\bigcap_{k=1}^n S_{\alpha_k} \right).$$

From the centeredness of the system δ follows the centeredness of the system $\zeta = \{T_\alpha\}_{\alpha \in A}$. Since dY is bicomact, ζ has a cluster point in dY . We prove that this point is unique.

1. Suppose that this is not so: $y_1 \in [\zeta], y_2 \in [\zeta], y_1 \neq y_2$. (By $[\zeta]$ we denote the set of all cluster points of the filter ζ .)
2. At the points y_1 and y_2 choose neighborhoods U_1 and U_2 with disjoint closures in d_{bX} .
3. $U_1 \cap Y$ and $U_2 \cap Y$ are far in the proximity space β .
4. Denote $W_1 = f^{-1}(U_1), W_2 = f^{-1}(U_2)$.
5. $W_1 \alpha W_2$, according to the definition of the proximity β .
6. Hence it follows that $[W_1]_{bX} \cap [W_2]_{bX} = \Lambda$.
7. $U_1 \cap T = \Lambda, U_2 \cap T = \Lambda$ for every $T \in \xi$.
8. Hence, and from the definition of the filter ξ , it follows that $W_1 \cap S = \Lambda, W_2 \cap S = \Lambda$ for every $S \in \delta$.
9. Since δ is the filter of all neighborhoods of the point x , $[W_1] \ni x, [W_2] \ni x$.
10. Items 9 and 6 contradict each other, as was required to prove.

Assign to each point $x \in X$ the unique limit point y of the filter $\zeta(x)$. The mapping g thus constructed will be the desired mapping $bX \rightarrow dY$.

§ 2. Decomposition of an irreducible mapping into elementary mappings. Let $f : X \rightarrow Y$ be an irreducible mapping of the bicomactum X onto the bicomactum Y . Let \mathfrak{B} be a base in X of cardinality τ ; U_1 and U_2 are elements of the base; $[U_1] \cap [U_2] = \Lambda$. Put

$$F(U_1, U_2) = f[U_1] \cap f[U_2].$$

Since the mapping f is irreducible, F is nowhere dense. $Y \setminus F$ is a locally bicomact space; it is a perfect irreducible image of its full preimage $P \subset X$.

By Theorem 2 there exists a bicomact extension $d_X(Y \setminus F)$ with the natural mapping

$$g : X \rightarrow d_X(Y \setminus F).$$

Lemma. $g(U_1) \cap g(U_2) = \Lambda$.

Proof of the lemma. The open sets $U_1 \cap f^{-1}(Y \setminus F)$ and $U_2 \cap f^{-1}(Y \setminus F)$ have disjoint closures in X . Their marked kernels are far in α ; consequently, the images V_1 and V_2 of these kernels in $Y \setminus F$ are far in the proximity space β . Hence it follows that

$$[V_1]_d \cap [V_2]_d = \Lambda,$$

where $d = d(Y \setminus F)$.

It is easy to verify that

$$g(U_i) \subseteq [V_i]_{d(Y \setminus F)} \quad (i = 1, 2),$$

as was required to prove.

Since \mathfrak{B} is a base of the bicomactum X of cardinality τ , for any two points x_1 and x_2 of X one can choose neighborhoods U_1 and U_2 with disjoint closures. It follows from the lemma that the mapping

$$g : X \rightarrow d_X[Y \setminus F(U_1, U_2)]$$

sends the points x_1 and x_2 to distinct points r_1 and r_2 in the space d_X .

Let η be the system of all sets $F(U_1, U_2)$, where $U_1, U_2 \in \mathfrak{B}$, and let ϑ be the system of all finite unions of sets from η . Obviously, the cardinality of the system ϑ is equal to τ . The system of all extensions $d_X(Y \setminus P)$ ($P \in \vartheta$) forms an inverse spectrum $\Sigma(\mathfrak{B})$ with natural projections

$$d_X(Y \setminus P_1) \rightarrow d_X(Y \setminus P_2),$$

when $P_1 \subseteq P_2$. (For the construction of the spectrum see (2).) Denote the limit of the spectrum $\Sigma(\mathfrak{B})$ by Y .

Theorem 3. $X = \lim \Sigma(\mathfrak{B})$.

Proof. Consider the system of mappings

$$\{g_\lambda : X \rightarrow d_X(Y \setminus F_\lambda)\}.$$

It generates a mapping

$$\{g : X \rightarrow d_X(Y \setminus F_\lambda)\}.$$

For any two points of X there is a mapping that sends them to distinct points in $d_X(Y \setminus F_{\lambda(x_1, x_2)})$. From the definition of the limit of a spectrum it follows at once that $\tilde{g}(x_1) \neq \tilde{g}(x_2)$. This means that the mapping g is one-to-one, i.e. a homeomorphism, as was required to prove.

From Theorem 3 it follows:

Theorem 4. *An irreducible mapping of a bicom pactum X of weight τ onto a bicom pactum Y decomposes into a spectrum of τ elementary mappings.*

If X and Y are compacta, we obtain:

Theorem 5. *An irreducible mapping of a compactum X onto a compactum Y is the product of a countable number of elementary mappings.*

Corollary of Theorem 5. *If $X \rightarrow Y$ is an irreducible mapping of a compactum X onto a compactum Y , then in Y there is everywhere a dense set of type G_δ of points of one-to-oneness.*

§ 3. Decomposition of an irreducible mapping of a bicom pactum into a transfinite spectrum of elementary mappings. Let $f : X \rightarrow Y$ be an irreducible mapping of a bicom pactum X onto a bicom pactum Y . Analogously to how this was done in ⁽²⁾, choose in Y an open everywhere dense set U so that $Y_1 = d_{XU}$ differs from Y . In Y choose an everywhere dense open set U_1 and construct $d_{XU_1} = Y_2$, and so on, by transfinite induction. If α is a limit ordinal, then as Y_α we take the limit of the spectrum of all Y_β for ordinals $\beta < \alpha$.

The transfinite process of constructing the spaces Y_α necessarily terminates at some cardinal number $\Omega(X, Y)$, since the weight of the bicom pactum X does not exceed 2^τ , where τ is the weight of the bicom pactum Y . Obviously, the limit of the spectrum of the spaces $\{Y_\alpha\}_{\alpha \leq \Omega}$ is an irreducible image of the bicom pactum X . It coincides with X , since otherwise one could find an open everywhere dense set $\tilde{U} \subset \tilde{Y}$ with the property $d_X \tilde{U} \neq \tilde{Y}$. This contradicts the fact that the process terminates at the space \tilde{Y} .

We have proved the following proposition:

Theorem 6. *Every irreducible mapping of a bicom pactum X onto a bicom pactum Y decomposes into a transfinite spectrum of mappings $\pi_{\alpha_2}^{\alpha_1}$, in which every mapping $\pi_{\alpha}^{\alpha+1}$ is elementary.*

§ 4. Incomplete absolutes. Let X be a bicom pactum and let η be some system of its closed nowhere dense subsets, possessing the following properties: 1) if $F_1 \in \eta$, $F_2 \in \eta$, then $F_1 \cup F_2 \in \eta$; 2) if $F_1 \subset F_2$, $F_2 \in \eta$, then $F_1 \in \eta$. We shall call the space X , together with the system η specified in it, a pair.

Let $f : Z \rightarrow X$ be an irreducible mapping, and suppose that for any two open sets U_1 and U_2 in Z whose closures are disjoint, one has

$$f[U_1] \cap f[U_2] \in \eta.$$

We shall say that the mapping f is generated by the system η . If $F \in \eta$, then $f^{-1}(F)$ is nowhere dense in Z . The mapping f and the system η generate in Z the system

$$\zeta = f^{-1}(\eta) = \{f^{-1}(F) \setminus F \in \eta\}.$$

Suppose that, along with the mapping f , there exists a mapping $g : Z \rightarrow Y$, and moreover $g^{-1}(g[U_1] \cap g[U_2]) \in \zeta$ ($[U_1] \cap [U_2] = \Lambda$). The mapping g generates in the space Y the system $\vartheta = \{g(F), F \in \zeta\}$. We shall say that the pair (X, η) is coabsolute with the pair (Y, ϑ) . It is easy to see that the mapping g is generated by the system ϑ .

Assertion. *The relation of coabsoluteness is an equivalence relation.*

Reflexivity and symmetry are obvious. Let us verify transitivity. Suppose $X(\eta) \sim Y(\zeta)$, $Y(\zeta) \sim Z(\vartheta)$. This means that there exist four mappings

$$P \rightarrow X, \quad Q \rightarrow Z, \quad P \rightarrow Y, \quad Q \rightarrow Y.$$

By Theorem 4, P and Q are limits of the spectrum of the spaces $b(Y \setminus F)$, $F \in \eta$. To each F there correspond two bicomact extensions $b_p(Y \setminus F)$ and $b_q(Y \setminus F)$.

Taking the extension $\beta(Y \setminus F)$ for each F and constructing a spectrum from them, we obtain in the limit a space R , which establishes a connection between X and Z by means of the natural mappings $R \rightarrow P \rightarrow X$, $R \rightarrow Q \rightarrow Z$.

Thus, with the aid of the pair (X, η) we construct a whole class of pairs $\{(Y, \zeta)\}$, and this class K is generated by any one of its pairs (Y, ζ) .

Let $Z, Y \in K$, and suppose there exists an irreducible mapping $f : Z \rightarrow Y$. Among all irreducible mappings $Z \rightarrow Y$ we choose those that are generated by the system ζ . From all the selected mappings we construct the system α .

Considering together the spaces from the class K and all mappings from α , we obtain the category \overline{K} . Let X be an arbitrary space of this category. Put $\widehat{X}(\eta) = \lim\{\beta U\}$, $u = X \setminus F$, $F \in \eta$.

Lemma 1. If $f : Z \rightarrow X$ is the mapping generated by the system η , then $\widehat{Z}(f^{-1}(\eta)) = \widehat{X}(\eta)$, where $f^{-1}(\eta)$ is the system of all complete inverse images of sets from the system η .

Lemma 2. If X_1 and X_2 are spaces of the category \overline{K} , then $\widehat{X}_1 = \widehat{X}_2$.

With the aid of these lemmas it is easy to prove

Theorem 7. \widehat{X} is a projective object in the category \overline{K} .

By passing to bicomact extensions, all the results obtained can be transferred to arbitrary completely regular spaces.

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Note: Figure translations are in progress. See original paper for figures.

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