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Abstract

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THE CHARACTER OF THE INSTABILITY OF THE INTERFACE BETWEEN TWO LIQUIDS IN A CONSTANT FIELD

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A plane interface between two liquids differing in their electric or magnetic properties becomes unstable in a sufficiently strong field parallel to the gravitational field ([1–3]). Depending on the value of the relative permeability of the media, both "soft" and "hard" regimes of excitation of the instability are possible.

The equilibrium shape of the interface between two liquid dielectrics in a constant electric field is determined by the condition of balance of the forces acting on the surface:

$$\left(\sigma_{ik}^{(2)} - \sigma_{ik}^{(1)}\right) n_k = \frac{\alpha}{R} n_i, \qquad \sigma_{ik} = -p\delta_{ik} - \frac{E^2}{8\pi}\left[\varepsilon - \rho\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T\right]\delta_{ik} + \frac{E_i D_k}{4\pi}. \quad (1)$$

Here σ_{ik} is the stress tensor ([4]); p is the pressure that would exist in the absence of the field; α is the surface-tension coefficient; $\mathbf{n}$ is the vector of the normal directed from the first medium into the second. We shall consider two-dimensional perturbations $z = \zeta(x)$ of an initially plane surface $z = 0$. Then in Laplace' s formula, which determines the surface pressure, there remains one radius of curvature $R = -(1 + \zeta'^2)^{3/2}/\zeta''$.

For the tangential components, equation (1) is satisfied identically by virtue of the boundary conditions of continuity of D_n and E_t:

$$\varepsilon_1 E_{1n} = \varepsilon_2 E_{2n}, \qquad E_{1\tau} = E_{2\tau}. \quad (2)$$

The normal component of (1), together with the condition of absence of body forces ([4])

$$\mathbf{f} = \nabla\left[-p + \frac{\rho}{8\pi}\left(\frac{\partial \varepsilon}{\partial \rho}\right)_T E^2 - \rho g z\right] = 0$$

leads to the equation

$$\text{const} - (\rho_1 - \rho_2) g\zeta + \frac{(\varepsilon_1 - \varepsilon_2)}{8\pi} E_1^2 + \frac{(\varepsilon_1 - \varepsilon_2)^2}{8\pi\varepsilon_2} E_{1n}^2 = -\frac{\alpha\zeta''}{(1+\zeta'^2)^{3/2}}. \quad (3)$$

For $\zeta \neq 0$ the electric field ceases to be homogeneous: with the curvature of the surface there is associated an additional field $\mathbf{e}(x,z) = -\nabla\varphi(x,z)$, whose potential satisfies the equation

$$\Delta\varphi_i = 0 \qquad (i = 1, 2), \quad (4)$$

and vanishes as $z \to \pm\infty$. Separating out from the total field strengths $\mathbf{E}_1$ and $\mathbf{E}_2$ the homogeneous parts, we obtain, instead of (2) and (3), the equations

$$\varepsilon\frac{\partial\varphi_1}{\partial z} - \frac{\partial\varphi_2}{\partial z} = \zeta'\left(\varepsilon\frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_2}{\partial x}\right),$$

$$\frac{\partial\varphi_1}{\partial x} - \frac{\partial\varphi_2}{\partial x} = \frac{\varepsilon - 1}{\varepsilon}E\zeta' - \zeta'\left(\frac{\partial\varphi_1}{\partial z} - \frac{\partial\varphi_2}{\partial z}\right),$$

$$\begin{aligned}\text{const} + \frac{\varepsilon_2(\varepsilon-1)}{8\pi}\Bigg\{&\left(\frac{\partial\varphi_1}{\partial x}\right)^2 + \varepsilon\left(\frac{\partial\varphi_1}{\partial z}\right)^2 + 2E\frac{\partial\varphi_1}{\partial z} \\ &-\frac{2}{\varepsilon}(\varepsilon-1)E\zeta'\frac{\partial\varphi_1}{\partial x} - 2(\varepsilon-1)\zeta'\frac{\partial\varphi_1}{\partial x}\frac{\partial\varphi_1}{\partial z} - \frac{\varepsilon-1}{\varepsilon^2}E^2\zeta'^2 - \frac{2}{\varepsilon}(\varepsilon-1)E\zeta'^2\frac{\partial\varphi_1}{\partial z}\Bigg\} \\ &-(\rho_1-\rho_2)g\zeta + \alpha\zeta''\left(1 - \frac{3}{2}\zeta'^2\right) = 0.\end{aligned} \quad (5)$$

which must be satisfied at $z = \zeta(x)$. Here E is the homogeneous part of the field in the second medium, $\varepsilon \equiv \varepsilon_1/\varepsilon_2$. Since we have in mind solving the problem by the method of expansion in the amplitude of the surface curvature, in the last equation we have retained only terms no higher than third order in ζ.

The solution of the system of equations (4)–(5) is sought in the form

$$\begin{aligned}\zeta(x) &= a\cos kx + a^2\beta\cos 2kx + a^3\gamma\cos 3kx + \dots, \\ \varphi_i(x,z) &= aA_i\cos kx\cdot e^{-kz} + a^2B_i\cos 2kxe^{-2kz} + \\ &+a^3\left(C_{1i}\cos kxe^{-kz} + C_{3i}\cos 3kx\cdot e^{-3kx}\right) + \dots \\ E &= E_0 + a^2E_2 + \dots, \qquad k = k_0 + a^2k_2 + \dots.\end{aligned} \quad (6)$$

Replacing a by $-a$ in these formulas must lead to a solution that differs from (6) only by a shift along the x-axis by a half-period. Therefore the expansions of E and k contain only even powers of a.

In the linear approximation one determines the critical field E_0, above which the plane surface is unstable with respect to small perturbations, and the critical wavelength of the perturbation:

$$E_0^2 = 8\pi \frac{\varepsilon(\varepsilon+1)}{\varepsilon_2(\varepsilon-1)^2}\sqrt{(\rho_1-\rho_2)g\alpha}, \qquad k_0 = \sqrt{(\rho_1-\rho_2)g/\alpha}. \tag{7}$$

These results of the linear calculation coincide with those obtained earlier [1–3]. The character of the instability can be clarified only in the third approximation, where E_2 is calculated:

$$E_2 = E_0 k_0^2 F^{-1}(\varepsilon), \qquad F(\varepsilon) = \frac{32(\varepsilon+1)^2}{142\varepsilon - 11(\varepsilon^2+1)}. \tag{8}$$

k_2 is not determined in this approximation. Substituting E_2 into the expansion for E, we obtain

$$ak_0 = \sqrt{\frac{F(\varepsilon)}{E_0}(E-E_0)}. \tag{9}$$

The character of the instability is determined by the sign of the function $F(\varepsilon)$. This quantity is positive for $\varepsilon_*^{-1} < \varepsilon < \varepsilon_*$, where $\varepsilon_* \approx 3.54$. For such values of the relative permittivity, a soft regime of excitation of the instability should be observed: near the critical E_0, the amplitude of the surface curvature grows in proportion to $\sqrt{E-E_0}$ (Landau law).

For $\varepsilon > \varepsilon_*$ the instability is hard. A hard regime of excitation should occur in all cases in which one of the media is a conductor. The electric field does not penetrate into the conducting medium, and therefore in formulas (7)–(9) one must put $\varepsilon \to \infty$.

The results of the analysis carried out also apply to the stability of the interface between ferromagnetic and ordinary liquids; it is only necessary everywhere to replace the electric field E by the magnetic field H, and the dielectric permittivity ε by the magnetic permeability μ.

An experimental study of the stability of the interface between media with $\mu < \mu_*$ was carried out in work [3]. Although the amplitude of the perturbations of the plane surface of the ferromagnetic suspension was not measured in these experiments, from the results presented in [3] it is clear that a soft instability occurred there. A hard instability of the water–air interface in an electric field was observed in the experiments [5].

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Note: Figure translations are in progress. See original paper for figures.