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S. G. SAMKO

A GENERALIZED ABEL EQUATION, THE FOURIER TRANSFORM, AND EQUATIONS OF CONVOLUTION TYPE

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We consider in $\mathcal{L}_p(-\infty, \infty)$ the integral equation of the first kind with a kernel of potential type (the generalized Abel equation)*:

$$M_\alpha \varphi \equiv \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{\infty} \frac{c_1(x) + c_2(x)\operatorname{sign}(x-t)}{|x-t|^{1-\alpha}} \varphi(t)\, dt = f(x), \tag{1}$$

where $c_1^2(x) + c_2^2(x) \neq 0,\ -\infty \leq x \leq \infty;\ 0 < \alpha < 1$. Denote by $I_+^\alpha \varphi,\ I_-^\alpha \varphi$ the fractional integrals, so that

$$M_\alpha \varphi \equiv u(x) I_+^\alpha \varphi + v(x) I_-^\alpha \varphi = f(x), \tag{2}$$

where $u(x) = c_1(x) + c_2(x),\ v(x) = c_1(x) - c_2(x)$.

Let us note that the usual inverse of the operator I_+^α,

$$\varphi(x) = I_+^{-\alpha} \Phi = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{\Phi(t)\, dt}{(x-t)^\alpha}, \tag{3}$$

which is valid for $\varphi(x) \in \mathcal{L}_1(-\infty, \infty)$, no longer holds for $\varphi(x) \in \mathcal{L}_p(-\infty, \infty)$, $p > 1$. Therefore, instead of the operator (3), one should consider its extension

$$I_+^{-\alpha} \Phi = \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{\Phi(x) - \Phi(t)}{(x-t)^{1+\alpha}}\, dt, \tag{4}$$

applicable in the range of the operator I_+^α, in particular for such $\Phi(x)$ that

$$\omega_p(\Phi; \delta) = O(\delta^\gamma), \qquad \gamma > \alpha, \tag{5}$$

where $\omega_p(\Phi;\delta)$ is the integral modulus of continuity of the function $\Phi(x)$. On the other hand, for $p=1$ the operator (3) is inverse, but another difficulty arises. The known methods of solving equation (1) in one way or another use the singular operator S, for which the space $\mathcal{L}_1(-\infty,\infty)$ is not invariant. However, in the case when the coefficients $c_1(x), c_2(x)$ are constant, the investigation in $\mathcal{L}_1(-\infty,\infty)$ can (^{8}) be carried through to the end and, in particular, a description of the range of the operator M_α can be given in terms of the absolute continuity of a certain auxiliary function.

In the present note, first, the range of the operator M_α is described in terms of the Fourier transform and an effective solution of the equation $M_\alpha\varphi = f$ is given; second, the equivalence of equation (1) to a certain equation of convolution type is established.

* The generalized Abel equation was first considered by K. D. Sakalyuk (^{6}) for the case of a finite interval $a < t,\ x < b$.

§ 1. Let us introduce notation

$$S\varphi = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\varphi(t)\,dt}{t-x}, \qquad P = \frac{p}{1-\alpha p},$$

$$\varkappa = \frac{1}{\pi}\int_{-\infty}^{\infty} d\arg\left\{c_1(x) - ic_2(x)\operatorname{tg}\frac{\alpha\pi}{2}\right\}.$$

By $H^\lambda(-\infty,\infty)$ we denote the class of functions $h(x)$ such that the functions

$$h\left(i\,\frac{t+1}{t-1}\right)$$

satisfy on the circle $|t|=1$ a Hölder condition of order λ. Then

$$|h(x_1)-h(x_2)| \le A|x_1-x_2|^\lambda/(1+|x_1|^\lambda)(1+|x_2|^\lambda).$$

Let also $\overline{\varphi}(x) = F(\varphi)(x)$ be the Fourier transform of the function $\varphi(x)$. Denote by $F_p(-\infty,\infty)$ the class of functions which are Fourier transforms of functions from $\mathcal{L}_{p'}(-\infty,\infty)$, $p' = p(p-1)^{-1}$, $p>1$.

By the Young–Hausdorff theorem $F_p(-\infty,\infty) = \mathcal{L}_p(-\infty,\infty)$ for $1<p\le 2$ and $F_p(-\infty,\infty)\subset\mathcal{L}_p(-\infty,\infty)$ for $p>2$. And, finally, by $R(-\infty,\infty)$ we denote the Wiener ring of Fourier transforms of functions from $\mathcal{L}_1(-\infty,\infty)$.

Theorem 1. Let $c_1(x), c_2(x)\in H^\lambda(-\infty,\infty)$, where $\lambda>\alpha$, and let $\varkappa\ge 0$. In order that the function $f(x)$ be representable in the form $f(x) = M_\alpha\varphi$, where $\varphi(x)\in\mathcal{L}_p(-\infty,\infty)$, $1<p<1/\alpha$, it is necessary and sufficient that

$$\text{for } p=2 \qquad |x|^\alpha\hat{f}(x)\in\mathcal{L}_2(-\infty,\infty), \tag{6'}$$

$$\text{for } 1 < p < 2 \qquad |x|^{\alpha} \hat{f}(x) \in F_{p'}(-\infty, \infty), \tag{6”}$$

$$\text{for } p > 2 \text{ it is sufficient that} \qquad |x|^{\alpha} \hat{f}(x) \in \mathcal{L}_{p'}(-\infty, \infty). \tag{6”’}$$

When conditions $(6')$–$(6''')$ are fulfilled, all solutions of equation (1) are determined from Abel’ s equation:

$$I_{+}^{\alpha} \varphi = \Phi(x), \tag{7}$$

where $\Phi(x)$ are solutions of the equation with Cauchy kernel from the class $\mathcal{L}_{p}(-\infty, \infty)$:

$$a_1(x)\Phi(x) + a_2(x)S\Phi = f(x), \tag{8}$$

$$a_1(x) = 2c_1(x)\cos^2(\alpha\pi/2) + 2c_2(x)\sin^2(\alpha\pi/2),$$

$$a_2(x) = [c_1(x) - c_2(x)]\sin\alpha\pi.$$

Let us add further that the solutions of equation (7) are found by formula (4). If, however, it is known that all solutions $\Phi(x)$ of equation (8) satisfy the condition

$$(\operatorname{ctg}(\alpha\pi/2) - i \operatorname{sign} x)\, |x|^{\alpha} \hat{\Phi}(x) \in R(-\infty, \infty), \tag{9}$$

then the solutions $\varphi(x)$ are also found by formula (3).

The proof of the theorem is based on the following lemmas:

Lemma 1*. The fractional integration operators $I_{\pm}^{\alpha}$ are equivalent to the operator of division by x^{α}. Namely, let $\varphi(x) \in \mathcal{L}_{p}(-\infty, \infty)$, $1 \le p \le 2$, and $p\alpha < 1$. Then

$$F(I_{\pm}^{\alpha}\varphi)(x) = \frac{e^{\pm i \frac{\alpha\pi}{2} \operatorname{sign} x}}{|x|^{\alpha}} F(\varphi)(x). \tag{10}$$

Equality (10) is also true for $2 < p < \alpha^{-1}$ (for $\alpha < 1/2$), if one additionally assumes that $F(\varphi) \in \mathcal{L}_{p'}(-\infty, \infty)$.

* Lemma 1 follows from an analogous result of G. O. Okikiolu (^{1}) for the Riesz potential; see also (^{4}).

The spaces $I_+^\alpha(\mathcal{L}_p)$, $I_-^\alpha(\mathcal{L}_p)$ of functions $f(x)$, representable in the form $f(x) = I_+^\alpha\varphi$ or $f(x) = I_-^\alpha\varphi$, respectively, where $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$, coincide, and we shall denote them simply by $I^\alpha(\mathcal{L}_p)$, $1 < p < 1/\alpha$.

Corollary to Lemma 1. In order that $f(x) \in I_+^\alpha(\mathcal{L}_p)$, for $1 < p \leqslant 2$ it is necessary and sufficient that conditions $(6')$–$(6'')$ be satisfied; for $p > 2$ it is sufficient that condition $(6''')$ be satisfied for $f(x)$; and for $p = 1$ it is necessary and sufficient that (9) be satisfied.

Lemma 2. The class of functions $I^\alpha(\mathcal{L}_p)$, for $1 < p < \alpha^{-1}$, is invariant with respect to the operator S and with respect to multiplication by functions from $H^\lambda(-\infty,\infty)$ when $\lambda > \alpha$.

Corollary to Lemma 2. Let $f(x)$ satisfy condition $(6')$ or $(6'')$. Then the functions Sf, $h(x)f(x)$ also satisfy the same condition. If, however, $f(x)$ satisfies condition $(6''')$, then one can only assert that $h(x)f(x)$ and Sf belong to $I^\alpha(\mathcal{L}_p)$.

Lemma 3. Let $c_1(x), c_2(x) \in H^\lambda(-\infty,\infty)$. If $f(x) \in I^\alpha(\mathcal{L}_p)$, then all solutions $\Phi(x)$ from $\mathcal{L}_p(-\infty,\infty)$ of equation (8) also belong to $I^\alpha(\mathcal{L}_p)$.

Equation (8), and in general Theorem 1, we obtain by using the following identities for $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$:

$$I_-^\alpha\varphi = \cos(\alpha\pi)\, I_+^\alpha\varphi + \sin(\alpha\pi)\, SI_+^\alpha\varphi, \qquad 1 \leqslant p < 1/\alpha,$$

$$SI_\pm^\alpha\varphi = I_\pm^\alpha S\varphi, \qquad 1 < p < 1/\alpha$$

(see ([2,7,8])).

Comparing Theorem 1 and Lemmas 2, 3, we conclude that the ranges of the operators $I_\pm^\alpha$ and M_α coincide:

$$I^\alpha(\mathcal{L}_p) = M_\alpha(\mathcal{L}_p), \qquad 1 < p < 1/\alpha,$$

provided that $c_1(x), c_2(x) \in H^\lambda(-\infty,\infty)$, $\lambda > \alpha$, and $\varkappa \geqslant 0$.

§ 2. Formulas (8) make it possible to establish a connection between equation (1) and equations of convolution type. We shall assume that

$$1 < p < 2/(1+\alpha), \tag{11}$$

It is known (([3]), p. 147) that $\widehat{\varphi}(x)|x|^{1/p'-1/r} \in \mathcal{L}_r(-\infty,\infty)$ for $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$ and $p \leqslant r \leqslant p'$. In particular,

$$\widehat{\varphi}(x)|x|^{-\alpha} \in \mathcal{L}_{P'}(-\infty,\infty); \tag{12}$$

the requirement $p \leqslant P' \leqslant p'$ is fulfilled by virtue of condition (11). Consequently, the functions $I_{\pm}^{\alpha}\varphi$ and

$$e^{\pm i\frac{\alpha\pi}{2}\operatorname{sign} x}\,|x|^{-\alpha}\widehat{\varphi}(x)$$

form (with the corresponding choice of signs) pairs of Fourier transforms from the spaces $\mathcal{L}_p(-\infty,\infty)$ and $\mathcal{L}_{P'}(-\infty,\infty)$, respectively, and we arrive at the following theorem on convolutions. Let $K(x) = \hat{k}(x)$.

Theorem 2. If $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$, where p satisfies condition (11), then

$$F(K(x)I_{\pm}^{\alpha}\varphi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k(t-x)\,\frac{\widehat{\varphi}(t)}{|t|^{\alpha}}e^{\pm i\frac{\alpha\pi}{2}\operatorname{sign} t}\,dt \tag{13}$$

provided one of the conditions is fulfilled:

$$1) \qquad k(x) \in \mathcal{L}_1(-\infty,\infty); \tag{14}$$

$$2) \qquad k(x) \in \mathcal{L}_P(-\infty,\infty), \qquad K(x) \in \mathcal{L}_{P'}(-\infty,\infty); \tag{15}$$

$$3) \qquad k(x) \in \mathcal{L}_r(-\infty,\infty), \qquad K(x) \in \mathcal{L}_{r'}(-\infty,\infty), \qquad r < P. \tag{16}$$

In case (14), both sides of equality (13) belong to $\mathcal{L}_{P'}(-\infty,\infty)$; in case (16), to $\mathcal{L}_Q(-\infty,\infty)$, $Q = Pr(P-r)^{-1}$; and in case (15) one can only assert that they exist almost everywhere.

Let now the coefficients $c_1(x), c_2(x)$ of equation (1) have the form

$$c_j(x) = \lambda_j + K_j(x), \qquad K_j(x) = \hat{k}_j(x), \qquad j = 1, 2,$$

where $k_j(x)$ satisfies one of the conditions (14)–(16). On the basis of Theorem 2, equation (1)–(2) is transformed into the form

$$(d_1 + d_2\operatorname{sign} x)\Phi_\alpha(x) + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}[m_1(x-t) + m_2(x-t)\operatorname{sign} t]\Phi_\alpha(t)\,dt =$$

$$= F(x), \tag{17}$$

where

$$d_1 = 2\lambda_1\cos(\alpha\pi/2), \quad d_2 = 2\lambda_2 i\sin(\alpha\pi/2), \quad F(x) = \hat{f}(x),$$

$$m_1(x) = 2\cos(\alpha\pi/2)k_1(-x), \quad m_2(x) = 2i\sin(\alpha\pi/2)k_2(-x), \quad \Phi_\alpha(x) = \hat{\varphi}(x)/|x|^\alpha.$$

A convolution-type equation of the form (17) in the space $\mathcal{L}_2(-\infty,\infty)$ was investigated by Yu. I. Cherskii (^{5}).

Equations (1) and (17) are equivalent in the sense that to every solution $\varphi(x)$ of equation (1) there corresponds, by the formula $\Phi_\alpha(x) = |x|^{-\alpha}\hat{\varphi}(x)$, a solution of equation (17), and, conversely, to every solution $\Phi_\alpha(x)$ of equation (17), representable in the form $\Phi_\alpha(x) = |x|^{-\alpha}F(\varphi)(x)$, where $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$, there corresponds a solution $\varphi(x)$ of equation (1). The solutions $\Phi_\alpha(x)$ of equation (17), therefore, must be sought in the space $\mathcal{L}_p(-\infty,\infty)$ or in the space with weight $\mathcal{L}'_p(|x|^{\alpha p'})$, and among these solutions one must choose those representable in the form $|x|^{-\alpha}\hat{\varphi}(x)$, $\varphi(x) \in \mathcal{L}_p(-\infty,\infty)$.

In particular, for the case of constant coefficients c_1, c_2 we obtain the following inversion formula:

$$\varphi(x) = \frac{1}{2}F^{-1}\left\{\frac{|x|^\alpha \hat{f}(x)}{c_1\cos(\alpha\pi/2) + ic_2 \operatorname{sign} x \sin(\alpha\pi/2)}\right\}.$$

For comparison we note that the summable solutions $\varphi(x) \in \mathcal{L}_1(-\infty,\infty)$ are given by the formula

$$\varphi(x) = \frac{1}{(c_1^2\cos^2(\alpha\pi/2) + c_2^2\sin^2(\alpha\pi/2))\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{c_2 + c_1\operatorname{sign}(x-t)}{|x-t|^\alpha}f(t)\,dt.$$

In conclusion we note that the connection between equations (1) and (17) may be of interest also for convolution-type equations themselves, since solving the latter in the space $\mathcal{L}_{p'}(|x|^{\alpha p'})$, $p' > 2/(1-\alpha)$, can be reduced* to solving the generalized Abel equation (1).

Rostov State University

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* It should, however, be stipulated that along this path one can find, generally speaking, not all solutions of a convolution-type equation, but only those of them which are representable in the form $|x|^{-\alpha}F(\varphi)$.

Note: Figure translations are in progress. See original paper for figures.