ON THE MATHEMATICAL DESCRIPTION OF ELEMENTS OF COMPLEX SYSTEMS

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Abstract

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ON THE MATHEMATICAL DESCRIPTION OF ELEMENTS OF COMPLEX SYSTEMS

The intensification of research on complex systems in engineering, organization of production, automation of control, economics, etc., stimulates the study of classes of systems described by known mathematical schemes: Boolean functions, finite automata, ordinary differential equations, queueing schemes, and others. At the same time, problems arise that take us beyond the limits of these classes of systems and require the use of unified schemes suitable for describing complex systems consisting of elements of different classes.

In the present article a mathematical scheme is proposed, called an **aggregate** ([1,2]). This scheme, as experience shows, is convenient for modeling many real complex systems.

Let Ω be some measurable space with probability measure μ ([3]). An **interrupted random process** will mean a pair $(\tau(\omega); z(t, \omega), 0 < t \leq \tau(\omega))$, where $\tau(\omega) \geq 0$, $z(t, \omega) \in Z$, $\omega \in \Omega$, and Z is an arbitrary set. A **random flow** will mean a pair $(I(\omega); Y_t(\omega), t \in I(\omega))$, where $I(\omega)$ is some flow of homogeneous events on the half-line $t \geq 0$, having with probability 1 no accumulation points other than $+\infty$, $Y_t(\omega) \in Y$, and Y is an arbitrary set.

To describe an aggregate, we shall need parametric families of interrupted random processes and random flows. The symbols denoting parameters will be written as arguments of the functions introduced above, before the symbol ω. An aggregate can receive input signals $x \in X$, send output signals $y \in Y$, and at any moment t from a prescribed interval possess an internal state $z(t) \in Z$. The sets X, Y, Z are called, respectively, the spaces of input signals, output signals, and (internal) states. The functioning of an aggregate proceeds as follows.

Let t_0 be a fixed initial moment of time; $(t_0) \leq t_1 < t_2 < \cdots < t_n < \cdots$ are the moments at which input signals $x_1, x_2, \ldots, x_n, \ldots$ arrive.

Assume that there is a measurable space Ω with two probability measures defined on it, $P_0(A)$ and $P(A)$, where A ranges over some Borel field of subsets of Ω; $\omega_0, \omega_1, \omega_2, \ldots, \omega_n, \ldots$ is a sequence of independent random elements of Ω, of which ω_0 follows the distribution P_0, while $\omega_1, \omega_2, \ldots$ follow the distribution P.

Consider the parametric family of interrupted random processes $(\tau(z,\theta,x,\omega);\xi(z,\theta,x,t,\omega);0<t\le\tau(z,\theta,x,\omega))$, where $\xi(z,\theta,x,t,\omega)\in Z$, depending on the parameters z,θ,x; $z\in Z$; $x\in X$; θ is a real number. Suppose that for some $n\ge 1$ we have $z(t_n)=z_n$ and $\bar{t}_n=\min\{t_{n+1},t_n+\tau(z_n,t_n,x_n,\omega_n)\}$. Then on the half-interval $(t_n,\bar{t}_n]$ the internal states of the aggregate are specified by the formula $z(t)=\xi(z_n,t_n,x_n,t-t_n,\omega_n)$.

If $(I(z,\theta,x,\omega);Y_t(z,\theta,x,\omega),t\in I(z,\theta,x,\omega))$ is a family of random flows depending on the same parameters, $Y_t(z,\theta,x,\omega)\in Y$, $t-t_n\in I(z_n,t_n,x_n,\omega_n)$ and $t-t_n\le\tau(z_n,t_n,x_n,\omega_n)$, then at time t the aggregate sends the output signal $y=Y_{t-t_n}(z_n,t_n,x_n,\omega_n)$.

Thus, the functioning of the aggregate for $t\geqslant t_1$ is completely determined. Let us now consider the interval $[t_0,t_1]$. Let $(\tau_0(t_0,\omega);\zeta_0(t_0,t,\omega),\ 0<t\leqslant\tau_0(t_0,\omega))$ be a terminating random process, and

$$\bar{t}_0=\min\{t_1,\ t_0+\tau_0(t_0,\omega_0)\}.$$

Then, for $t\in[t_0,\bar{t}_0]$, the internal state of the aggregate is determined by the formula

$$z(t)=\zeta_0(t_0,t-t_0,\omega_0).$$

Finally, suppose that a random flow is given

$$(I_0(t_0,\omega);\ Y_t(t_0,\omega),\ t\in I_0(t_0,\omega)),$$

where t_0 is a parameter. We shall assume that every time $t\in[t_0,\bar{t}_0]$ and $t-t_0\in I_0(t_0,\omega_0)$, the aggregate sends an output signal

$$y=Y_{t-t_0}(t_0,\omega_0).$$

To this we add the following convention. If, for some $n\geqslant 1$,

$$t_n+\tau(z_n,t_n,x_n,\omega_n)<t_{n+1},$$

or if

$$t_0+\tau_0(t_0,\omega_0)<t_1,$$

then for

$$t>t_n+\tau(z_n,t_n,x_n,\omega_n)$$

(respectively, for $t>t_0+\tau_0(t_0,\omega_0)$) neither $z(t)$ nor the output signals of the aggregate are defined. We shall also assume that the aggregate sends no other signals besides those formed in accordance with the rules indicated above.

In solving many practical problems it is expedient to classify in different ways the input signals arriving at the aggregate, depending on the character of the effect they exert on its functioning. The following construction, which develops the notion of an aggregate introduced above, appears useful. Here we shall denote the space of input signals, the space of states of the aggregate, and the elements of these spaces by the symbols introduced above, furnished with a bar.

Suppose that $\bar{X} = X \times \Gamma$, where X and Γ are certain sets; similarly, $\bar{Z} = Z \times \Gamma$, i.e.,

$$\bar{x} = (x, g),$$

where $x \in X,\ g \in \Gamma$;

$$\bar{z}(t) = (z(t), g(t)),$$

where $z(t) \in Z,\ g(t) \in \Gamma$; x and g may take the "empty" value ϕ. Then, with respect to the given representation of the space $\bar{Z}$ in the form of the direct product $Z \times \Gamma$, the signal

$$\bar{x} = (x, \phi)$$

is called an ordinary (input) signal, and

$$\bar{x} = (\phi, g)$$

a control signal.

The construction of complex systems from aggregates may require a finer classification of signals, taking into account different ranks or levels of control.

In the practical study of systems described by the scheme proposed above, the method of realizing the random processes and flows belonging to it is not without interest. In this connection, an aggregate may be represented as an object determined by the spaces $\bar{X}$, Z, $\bar{Y}$ and by random operators defined on $Z \times \bar{X}$: the transition operator (to a new state) H and the output operator G. The random operator H generates possible realizations of the process $z(t)$, and the random operator G realizations of flows of output signals.

Various methods of constructing these operators may be used. Let us consider one of them. Represent the operator G in the form of a collection of random operators G' and G'', of which the first generates the successive instants at which output signals are issued, and the second the contents of the signals. For this purpose, in the space Z define the set $Z^{(Y)}(g(t), \alpha)$ in such a way that, if for the given instant of time t the state

$$z(\theta) \subset Z^{(Y)}(g(\theta), \alpha)$$

for

$$t - \varepsilon < \theta < t,$$

where ε is an arbitrary positive number, and

$$z(t) \in Z^{(Y)}(g(t), \alpha),$$

then t is the instant of issuance of the output signal

$$y = G''[\bar{z}(t), \alpha] = G''[z(t), g(t), \alpha].$$

Here α is a parameter, $\alpha \in A$; A is the parameter space of the aggregate. The value

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_p)$$

is fixed within the framework of each particular problem; $\alpha_1, \alpha_2, \dots, \alpha_p$ are usually called constructive parameters. In this connection, $g(t)$ may be called a control parameter.

We shall represent the transition operator H in the form of a collection of random operators U, V', and V''. In addition to the state $z(t)$, we shall need the state $z(t+0)$, in order to distinguish the states of the aggregate before and after the instant at which an external signal arrives, as well as the extended state

$$z^* = (\tau, z).$$

Let t'_n be the instant at which the signal

$$\bar{x}'_n = (x'_n, \phi)$$

arrives at the aggregate; then

$$z^*(t'_n + 0) = V'[\bar{z}(t'_n), x'_n, t'_n, \alpha].$$

If t''_n is the instant at which the signal arrives at the aggregate—

of the signal $\bar{x}'' = (\phi, g''_n)$, then $z^*(t''_n + 0) = V''[z(t''_n), g''_n, t''_n, a]$. Further, when t_n is the moment of arrival at the aggregate of the signal $\bar{x}_n = (x_n, g_n)$, then $z^*(t_n + 0) = V'''[z(t_n), g_n, t_n, a]$, and $z^*(t_n + 0 + 0) = V'[z(t_n + 0), x_n, t_n, a]$. Finally, if $(t_n, \bar{t}_n]$, where $\bar{t}_n = \min\{t_{n+1}, t_n + \tau_n\}$, contains no moment of arrival of an external signal, then for $t \in (t_n, \bar{t}_n]$

$$z(t) = U[z(t_n + 0), t_n, a, t].$$

Let us show how to construct an aggregate describing the operation of a queuing system. The internal states of such an aggregate may be specified in the form

$$z(t) = (n_1, n_2, \dots, n_k;\ z_1, z_2, \dots, z_{i(n_1, n_2, \dots, n_k)}),$$

where n_k are discrete parameters (the number of requests in the system, the types of these requests, the state of the service devices, etc.), and z_i are continuous parameters. Here z_k has the meaning of the time that must be spent, beginning from the moment t, for the complete servicing of the k-th request. The input signals may be represented in the form $x = (l_1, \dots, l_s; x_1, \dots, x_{j(l_1, \dots, l_s)})$, where $l_1, \dots, l_s$ are discrete parameters indicating the type of the requests, and x_k has the meaning of the duration of the k-th kind of service of the given request. Thus an input signal is equivalent to the arrival of a request into the system. Analogously, an output signal is interpreted as the completion of service of some request. Obviously, $Z^{(Y)}$ can have the form only of a subset of the set

$$\bigcup_{n_1,\dots,n_k;\ i}\ \bigcup_{z_i}\{z(t)=(n_1,\dots,n_k;\ z_1,\dots,z_{i-1},0,z_{i+1},\dots)\}.$$

The operator H reduces to the fact that $n_1, n_2, \dots, n_k$ are changed and new coordinates z_j, equal to some of $x_1, x_2, \dots$, are also attached to the existing coordinates z_j.

The operator G is completely determined by specifying the set $Z^{(Y)}$ and a function on this set with values in Y.

Consider an aggregate for which Γ consists of a single element, X and Y are finite sets, the state space $Z = Z' \times Z_{m+1}$; $z' = (z_1, z_2, \dots, z_m) \in Z'$; Z' is a finite set; $t - [t] = z_{m+1} \in Z_{m+1}$, where (t) is the integer part of t. At the moment $t = 0$, $z'(t) = z'_0$; $\tau_0 = 1$. At the moments $t = 1, 2, \dots$ input signals $x(i) \in X$ enter the aggregate. The operator V' is specified by the relations $z'(i+0) = \phi[z'(i), x(i)]$; $\tau_i = 1$; the operator U by the relation $z'(t) = z'([t])$; the set $Z^{(Y)}$ by the relation $z_{m+1} = 0$; $t_k = i+1$; the operator G'' by the relation $y_k = \psi[z'(i), x(i)]$; ϕ and ψ are nonrandom functions (the space Ω consists of a single element). This aggregate at the points $t = i$ coincides with the finite automaton (^{4}), for which the input alphabet is X, the output alphabet is Y, the set of internal states is Z', the initial state is z'_0, the transition function is ϕ, and the output function is ψ.

If we now suppose that Γ is an arbitrary finite set, we obtain an automaton of variable structure, which changes under the influence of a control signal. The consideration of such automata is essential for the theory of reliability of computers.

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Note: Figure translations are in progress. See original paper for figures.