MEAN VALUES OF MULTIPLICATIVE FUNCTIONS

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Abstract

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MEAN VALUES OF MULTIPLICATIVE FUNCTIONS

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In works ($^{1-6}$) various asymptotic formulas were proved for sums of multiplicative functions, based on information about the behavior of these functions at prime numbers. In the present note we formulate several new results of this type and their applications in probabilistic number theory.

A modification of the method of work (^{6}) leads to the following theorem.

Theorem 1. Let $f(n)$ be a multiplicative function,

$$\sum_{p\le x}|f(p)| = O\left(\frac{x}{\ln x}\right), \qquad f(p^r) = O((2p)^{r\gamma}) \quad (r\ge 2,\ \gamma<1/2), \tag{1}$$

$$\sum_{p\le x}\frac{f(p)\ln p}{p} \sim \tau \ln x. \tag{2}$$

If there exists a numerical sequence $\{a_p\}$ such that

$$\sum_{p\le x}|a_p| = O\left(\frac{x}{\ln x}\right), \qquad \sum_{p\le x} a_p \sim \sigma\frac{x}{\ln x}, \qquad \sigma = \text{const}; \tag{3}$$

$$\lim_{y\to\infty}\sum_{y<p\le y^2}\frac{|f(p)|-|f(p)-a_p|}{p} > 0, \tag{4}$$

then

$$\sum_{n\le x} f(n) = \frac{e^{-C\tau}}{\Gamma(\tau)}\frac{x}{\ln x}\prod_{p\le x}\left(1+\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right) + o\left(\frac{x}{\ln x}\prod_{p\le x}\left(1+\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r}\right)\right), \tag{5}$$

where C is Euler’ s constant.

It is known (^{1}) that formula (5) does not follow from conditions (1) and (2) alone. Various additional restrictions in place of (4) were considered in works (^{2,6}). The following results are consequences of Theorem 1.

Theorem 2. Let $f(n)$ satisfy conditions (1) and (2) of the preceding theorem, and let E be a certain set of prime numbers such that

$$\sum_{\substack{p\leq x\\ p\in E}} 1 \sim \delta\frac{x}{\ln x}, \qquad \delta > 0. \tag{6}$$

Then, if the values $f(p)$ for $p \in E$ lie inside the angle

$$|\arg(z/z_0 - 1)| \leq \pi/2 - \varepsilon, \qquad \varepsilon > 0, \tag{7}$$

with vertex at the point $z_0 \neq 0$, then (5) holds.

In particular, formula (5) is valid if, for $p \in E$, $f(p) \geq c_1 > 0$.

Theorem 3. If $f(n)$ satisfies conditions (1) and (2) and, moreover,

$$\overline{\lim}_{p\to\infty}|f(p)| - \lim_{p\to\infty}|f(p)| < 2(\tau), \tag{8}$$

then (5) holds.

For condition (8) to hold, it is clearly sufficient that $|f(p)| = \text{const}$ and $t \neq 0$, or that

$$\overline{\lim_{p\to\infty}}|f(p)| < 2|\tau|. \tag{9}$$

For applications to probabilistic number theory, the following special case of Theorem 3 is important.

Theorem 4. Let $\psi(n)$ be a real additive function such that, as $x \to \infty$,

$$\frac{1}{\ln x}\sum_{\substack{p\leq x\\ \psi(p)<w}} \frac{\ln p}{p} \to F(w). \tag{10}$$

where $F(w)$ is a distribution function (in the sense of weak convergence). Then

$$\sum_{n\leq x} e^{i\xi\psi(n)} = \frac{e^{-C\tau(\xi)}}{\Gamma(\tau(\xi))}\frac{x}{\ln x}\prod_{p\leq x}\left(1 + \sum_{r=1}^{\infty}\frac{e^{i\xi\psi(p^r)}}{p^r}\right) + o(x) \tag{11}$$

uniformly in ξ in any finite interval, where $\tau(\xi)$ is the characteristic function of the distribution $F(w)$.

Previously, (11) could be obtained only from the stronger assumption

$$N(p \leq x,\ \psi(p) < w) \sim F(w)x/\ln x.$$

An immediate consequence of (10) is the well-known Erdős-Delange theorem (^{5}) on the distribution of values of additive functions.

In (^{3}) Delange proved that if $f(n)$ is multiplicative, $|f(n)| \leq 1$, and the series $\sum_p \frac{f(p)-1}{p}$ converges, then the mean value

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n)$$

exists. From Theorem 3 the following generalization of Delange' s theorem follows immediately.

Theorem 5. If $f(n)$ is multiplicative, $\tau > 0$, $|f(p)| \leq \tau$ for all p, $f(p^r) = O((2p)^{r\gamma})$ for $r \geq 2$, $\gamma < 1/2$, and the series

$$\sum_p \frac{f(p)-\tau}{p}$$

converges, then, as $x \to \infty$,

$$\sum_{n\leq x} f(n) \sim Ax(\ln x)^{\tau-1}, \qquad A = \text{const}.$$

For applications to questions of the uniform distribution of fractional parts of additive functions, the following theorem is of interest.

Theorem 6. If $f(n)$ is multiplicative, $|f(n)| \leq 1$, E is a set of prime numbers such that

$$\sum_{\substack{p\leq x\\ p\in E}} \frac{\ln p}{p} \sim \delta \ln x, \qquad \delta > 0,$$

$$\sum_{\substack{p\leq x\\ p\in E}} f(p) \sim \tau \frac{x}{\ln x},$$

and $\tau \neq \delta$, then

$$\sum_{n\leq x} f(n) = o(x).$$

In particular, if E is the set of all prime numbers, this gives another theorem of Delange (^{4}). We also note the following fact:

Theorem 7. If the fractional parts of a real additive function $\psi(m)$ are uniformly distributed when m runs through the numbers composed ···

of primes belonging to the set E, then they are uniformly distributed also when m runs through all natural numbers.

Starting from Theorem 4 and using the method of paper (7), one can generalize many results of probabilistic number theory. Thus, for example, the following holds:

Theorem 8. Let $\psi(m)$ be a real additive function, and let M be a set of primes such that

$$\sum_{p\in M} \frac{1}{p} < \infty,$$

and suppose that there exists at least one $p \in M$ such that $\psi(p) \neq 0$. Then, if

$$\max_{\substack{p\leq x\\ p\in M}} |\psi(p)| = o\left(\left(\sum_{\substack{p\leq x\\ p\in M}} \frac{\psi^2(p)}{p}\right)^{1/2}\right),$$

then

$$\frac{1}{n}N\left(m \leq n,\, \psi(m) - \sum_{\substack{p\leq n\\ p\in M}} \frac{\psi(p)}{p} \leq x\left(\sum_{\substack{p\leq n\\ p\in M}} \frac{\psi^2(p)}{p}\right)^{1/2}\right) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\, du.$$

This theorem generalizes the well-known result of Erdős-Kac (8), which is obtained when M is the set of all primes and $\psi(m)$ is bounded on the primes.

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