ON THE CONVERGENCE OF THE RAYLEIGH METHOD

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Abstract

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ON THE CONVERGENCE OF THE RAYLEIGH METHOD

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1. The problem of the stability of a plane-parallel flow of an ideal fluid reduces to the eigenvalue problem for the equation:

$$[U(y)-c](\varphi''-\alpha^2\varphi)-U''(y)\varphi=0 \tag{1,1}$$

with homogeneous boundary conditions of the form $\varphi(0)=0,\ \varphi(1)=0$.

Here all parameters are determined by the fact that the stream function for the disturbance is sought in the form $\psi=\varphi(y)\exp[i\alpha(x-ct)]$. The Rayleigh method consists in replacing the velocity profile of the basic flow by a piecewise-linear one: $U_n(y)=U_n(k/n)(y-k/n)$ for $k/n\le y\le(k+1)/n$; $U_n(0)=U(0)$. Thus, in equation (1,1) the term $\varphi U''(y)$ is discarded, which makes it possible to write the general solution

$$\varphi_n(y)=\sum_{m=1}^{k}a_m\operatorname{sh}\alpha[y-(m-1)/n]$$

for $(k-1)/n\le y\le k/n$. The constants a_k are chosen so that the boundary condition $\varphi_n(1)=0$ and the condition of pressure continuity [1] at $y=k/n$ $(k=1,2,\dots,n-1)$ are satisfied:

$$\alpha[U_n(k/n)-c]_{k+1}a=$$

$$=\left\{\sum_{m=1}^{k}a_m\operatorname{sh}\alpha[k/n-(m-1)/n]\right\}[U'(k/n)-U'((k-1)/n)].$$

For the proof of convergence of the approximate eigenvalues it is convenient to replace equation (1,1) by the system of equations:

$$\varphi'[U(y) - c] - \varphi U'(y) = p(y),$$

$$\varphi[U(y) - c] = p'(y). \tag{1,2}$$

Here $p(y)$ is a certain differentiable function. If we denote

$$\mathbf{g} = \{g_1, g_2\} = \{\varphi', \alpha\varphi\}, \quad \mathbf{f} = \{f_1, f_2\} = \{p, \frac{1}{\alpha}p'\},$$

then (1,2) can be written in the form:

$$c\mathbf{g} + \mathbf{f} = L_1(\mathbf{g}). \tag{1,3}$$

We note that

$$\alpha g_1 - g_2' = 0, \tag{1,4}$$

$$f_1' - \alpha f_2 = 0. \tag{1,5}$$

Consider the Hilbert space $\mathbf{L}_2(0,1)$ of vector-functions $\mathbf{g} = \{g_1; g_2\}$ with components g_k from $L_2(0,1)$. Denote by $\Gamma'(0,1)$ the set of infinitely differentiable finite on $[0;1]$ vectors satisfying (1,4), and by $\Gamma^0(0,1)$ its closure in the norm $\mathbf{L}_2(0,1)$. The orthogonal complement of the subspace $\Gamma^0(0,1)$ will be denoted by $G(0,1)$.

Lemma 1. $G(0,1)$ consists of vector-functions satisfying condition (1,5).

Proof. Let $\mathbf{f} \in G(0,1)$. Then for any $\mathbf{g} \in \Gamma'(0,1)$

$$\int_0^1 (f_1 g_1 + f_2 g_2)\,dy = 0.$$

Since $g_2 = \alpha \int_0^y g_1\,dy$ and $f_2 = \left(\int_0^y f_2\,dy\right)'$, we have

$$\int_0^1 (f_1 g_1^* + f_2 g_2^*)\,dy = \int_0^1 f_1 g_1^*\,dy + \int_0^1 \left(\frac{d}{dy}\int_0^y f_2\,dy \cdot \alpha \int_0^y g_1^*\,dy\right) dy =$$

$$= \int_1^1 \left[f_1 - \int_0^y f_2\,dy \cdot \alpha\right] g_1^*\,dy = 0.$$

By virtue of the sufficient arbitrariness of g_1^*, we obtain

$$f_1 - \alpha \int_0^y f_2 \, dy = \text{const},$$

which was to be proved.

It is not difficult to see that the equation $g - \frac{1}{c} f = \psi$ defines a bounded linear operator $g = L_2(\psi)$. Here ψ is prescribed in $L_2(0.1)$, g is sought in $\Gamma^0(0.1)$, and f is sought in $G(0.1)$. Consequently, solving equation (1.3) is equivalent to solving the equation $g = c^{-1} L_2(L_1(g)) = c^{-1} L(g)$, since $G(0.1)$ consists of vectors whose components satisfy equation (1.5). Similarly, the approximate eigenvalues are determined from the solution of the equation $g = c^{-1} L_n(g)$, where $\|L - L_n\| \to 0$ as $n \to \infty$ (L is a bounded linear operator). The proof of convergence now becomes quite obvious. We formulate the final result as a theorem.

Theorem 1. *If the closed bounded set D_0 contains no points of the spectrum of the operator L, then, beginning with some number n, D_0 contains no points of the spectrum of the operator L_k. Conversely, for any eigenvalue λ_0 of the operator L that is an isolated point of the spectrum, one can specify a sequence of approximate eigenvalues converging to λ_0.*

Using the foregoing, it is not difficult to prove the convergence of the modified Rayleigh method proposed in [2] for solving the problem of the linear development of perturbations of a nonstationary plane-parallel flow of an ideal fluid. The problem under consideration is equivalent to solving the partial differential equation:

$$[\partial/\partial t + iU][\partial^2/\partial y^2 - \alpha^2]\varphi - i\alpha\varphi \, \partial^2 U/\partial y^2 = 0 \tag{1.6}$$

with boundary conditions $\varphi(0, t) = 0$, $\varphi(1, t) = 0$, $\varphi(y, 0) = \varphi_0(y)$. The method consists in replacing the velocity profile $U(y, t)$ $(0 \le y \le 1)$ by profiles $U_n(y)$ that are piecewise linear in y:

$$U_n = U_n(k/n; t) + [y - k/n] \partial U(k/n; t)/\partial y \quad \text{for} \quad k/n < y < (k+1)/n.$$

For $k/n \le y \le (k+1)/n$, the approximate expression for the perturbation stream function is represented in the form

$$\psi_n = \varphi_n(y, t) \exp[i\alpha(x)],$$

where

$$\varphi_n(y,t) = \sum_{i=1}^{k} a_i(t)\,\mathrm{sh}\,\alpha[y-(i-1)/n] \quad \text{for} \quad (k-1)/n \le y \le k/n.$$

From the conditions of pressure continuity and the vanishing of the normal component of the velocity at $y = 1$, one obtains a system of equations for the unknowns $a_k(t)$. The initial data for $a_k(t)$ are determined from the known stream function of the initial perturbation distribution.

The proof of convergence is based on the fact that uniform estimates can be obtained for the sequence of approximate solutions, since the approximate φ_n and the exact solution φ satisfy "almost" the same system of differential equations:

$$\partial^2\varphi/\partial y\,\partial t + U(y,t)i\alpha\,\partial\varphi/\partial y - i\alpha\varphi\,\partial U/\partial y = i\alpha P(y,t),$$

$$-i\alpha\,\partial\varphi/\partial t + U(y,t)\alpha^2\varphi = \partial P(y,t)/\partial y \tag{1.7}$$

and, moreover, the necessary continuity conditions are satisfied. (For an approximate solution the coefficients U and $\partial U/\partial y$ in system (1.7) are replaced by U_n and $\partial U_n/\partial y$.)

Consider the spaces of vector functions $L_2(Q_T)$, $\Gamma'(Q_T)$, $\Gamma^0(Q_T)$, $G(Q_T)$, which differ from the corresponding spaces introduced earlier only in that all vector functions now depend on y and on t; $0 \le t \le T$, $0 \le y \le 1$. It is assumed that for almost all t the elements $G(Q_T)$ belong to $G(0,1)$ and the elements $\Gamma^0(Q_T)$ to the space $\Gamma^0(0,1)$; $Q_T = [0,T] \times [0,1]$.

By a generalized solution of problem (1.6) we shall mean a vector function $g \in \Gamma^0(Q_T)$, for which the derivative $\partial g/\partial t$ exists and is quadratically summable over Q_T, and g satisfies the conditions $g(y,0) = \{\varphi_0'(y); \alpha\varphi_0(y)\}$, $g_2|_{y=0} = 0$, $g_2|_{y=1} = 0$, and the identity

$$\int_0^T \int_0^1 \left(\frac{\partial g}{\partial t} + iU\alpha g - ig_2\frac{\partial U}{\partial y}\right)\Phi(y,t)\,dy\,dt = 0 \tag{1.8}$$

for all $\Phi \in \Gamma'(Q_T)$ $(U = \{U(y,t), 0\})$.

If problem (1.6) has a generalized solution, then this solution satisfies equations (1.7) almost everywhere by Lemma 1. The uniqueness of the generalized solution is obvious.

Using (1.7) and denoting

$$z = \int_0^y g_m g_m^*\,dy,$$

we obtain $dz/dt \leq \text{const} \cdot z$, i.e., the uniform boundedness of the sequence of approximate solutions $\{g_n\}$. Similarly, $\{\partial g_n/\partial t\}$ is estimated. Since $L_2(Q_T)$ is weakly compact, one can select $\{g_m\}$ such that $\{g_m\}$ and $\left\{\frac{\partial}{\partial t} g_m\right\}$ converge weakly to g and g_t, respectively, and g satisfies (1.8).

2. The stability of a plane rectilinear flow of a viscous incompressible fluid with respect to infinitesimal disturbances is determined by the eigenvalues of the following problem:

$$(U-c)\varphi' - U'\varphi + \frac{i}{\alpha\,\mathrm{Re}}(\varphi''' - \alpha^2\varphi') = p(y),$$

$$\alpha^2(U-c)\varphi + \frac{i}{\alpha\,\mathrm{Re}}(\alpha^2\varphi'' - \alpha^4\varphi) = p'(y),$$

$$\varphi(0) = 0, \qquad \varphi'(0) = 0, \qquad \varphi(1) = 0, \qquad \varphi'(1) = 0. \tag{2.1}$$

Here $\mathbf{V} = e^{i\alpha(x-ct)}\{\varphi', -\alpha\varphi\}$ is the velocity vector of the disturbance, $U(y)$ is the velocity distribution in the parallel flow, $p = p(y)\exp[i\alpha(x-ct)]$ is the pressure variation associated with the disturbance, and Re is the Reynolds number.

The method for the approximate determination of the eigenvalues consists in replacing the velocity profile $U(y)$ as before by a piecewise-linear* profile $U_n(y)$, and seeking the approximate solution φ_n in the form $\sum_{i=1}^{4} a_{ki} g_{ki}(y)$ for $k/n \leq y \leq (k+1)/n$, $k = 0, 1, \dots, n-1$, where $g_{ki}(y, \alpha, \mathrm{Re}, c)$ are determined from the conditions:

$$[U_n - c]\,[g''_{ki} - \alpha^2 g_{ki}] + \frac{i}{\alpha\,\mathrm{Re}}\,(g''''_{ki} - 2\alpha^2 g''_{ki} + \alpha^4 g_{ki}) = 0,$$

$$g^{(j)}_{ki}(k/n) = \begin{cases} 0, & j \neq i-1, \\ 1, & j = i-1; \end{cases} \qquad j = 0, 1, 2, 3. \tag{2.2}$$

* For carrying out direct computations it is convenient to replace $U(y)$ and $U'(y)$ by piecewise-constant functions. For the proof of convergence this change is immaterial.

The conditions of continuity of the pressure, the velocity, the first derivative of the velocity, and the vanishing of the velocity at $y = 1$ and $y = 0$ lead to a homogeneous system of linear algebraic equations with respect to the constants a_{ki}. This system has a nontrivial solution if the determinant of the system $\Delta_n(\alpha, \mathrm{Re}, c)$ is equal to zero.

Theorem 2. *The eigenvalues of problem (2.1) are obtained by passing to the limit, as $n \to \infty$, from the equation $\Delta_n(\alpha, \mathrm{Re}, c) = 0$.*

The proof is based on reducing problem (2.1) to the problem of determining the characteristic numbers λ of an equation of the form $\mathbf{g} = \lambda T(\mathbf{g})$ in the Hilbert space $\mathbf{L}_2(0,1)$, where T is a completely continuous operator (the notation was introduced above).

Consider the equation

$$\mathbf{g}'' - \alpha^2 \mathbf{g} + \mathbf{f} = \psi, \quad \psi = \{\psi_1, \psi_2\}. \tag{2.3}$$

Here $\mathbf{g}$ is sought in $\mathbf{I}^0(0,1)$, $\mathbf{f}$ is in $\mathbf{G}(0,1)$, and $\psi \in \mathbf{L}_2(0,1)$. By a generalized solution of (2.3) is meant a vector function $\mathbf{g} \in \mathbf{I}^0(0,1)$ such that, for every Φ from $\mathbf{I}(0,1)$,

$$\int_0^1 \left[\mathbf{g}'(\Phi^*)' + (\alpha^2 \mathbf{g} + \psi)\Phi^*\right] dy = 0.$$

If one introduces the space $\mathbf{H}$, obtained by completing $\mathbf{I}(0,1)$ in the norm corresponding to the scalar product

$$[\mathbf{U}, \mathbf{V}] = \int_0^1 \mathbf{U}'(\mathbf{V}^*)'\, dy +$$

$$+\alpha^2 \int \mathbf{U}\mathbf{V}^*\, dy,$$

then the preceding condition takes the form $[\mathbf{g}, \Phi]$. Since $-(\psi, \Phi)$ defines a linear functional on Φ, there exists a vector $\mathbf{g}_0$ such that $-(\psi, \Phi) = (\mathbf{g}_0, \Phi)$. From this the unique solvability of (2.3) follows easily. Thus, equation (2.3) defines an operator $M(\psi) = \mathbf{g}$ (since different $\mathbf{g}$' s correspond to different ψ' s). If $\psi \in \mathbf{L}_2(0,1)$, then $\mathbf{g} \in \mathbf{H}(0,1)$ and

$$\|\mathbf{g}\|_H \leq \text{const} \cdot \|\psi\|_{L_2}. \tag{2.4}$$

From (2.4) it follows that the operator M is completely continuous, since a bounded set in $\mathbf{H}$ is compact in $\mathbf{L}_2(0,1)$.

Thus, we have proved

Lemma 2. *The operator M, with domain $\mathbf{L}_2(0,1)$, is completely continuous.*

Rewrite (2.1) in the form

$$\mathbf{g}'' - \alpha^2 \mathbf{g} + \mathbf{f} = \text{Re}\ L(\mathbf{g}) \quad (\mathbf{f} = i\alpha\ \text{Re}\{p; p'/\alpha\}). \tag{2.5}$$

Here L is a bounded operator. By a solution of problem (2.5) one may mean a vector function from $\mathbf{I}^0(0,1)$ satisfying the equation

$$\mathbf{g} = \lambda M(L(\mathbf{g})) \equiv \lambda T(\mathbf{g}). \tag{2.6}$$

Here $\lambda = R$, and T is a completely continuous operator in $\mathbf{L}_2(0,1)$. If U is replaced by U_n, and U is a sufficiently smooth function, then T is replaced by a completely continuous operator T_n, and $\|T - T_n\| \to 0$ as $n \to \infty$. As is known, in this case the characteristic numbers λ of equation (2.6) are obtained by passing to the limit, as $n \to \infty$, from the characteristic numbers of the equation $\mathbf{g} = \lambda T_n(\mathbf{g})$.

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Note: Figure translations are in progress. See original paper for figures.