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Abstract

Full Text

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ON THE NUMERICAL INTEGRATION OF CERTAIN SYSTEMS IN THE THEORY OF NONLINEAR OSCILLATIONS

(Presented by Academician V. I. Smirnov, 26 IX 1968)

In the present note we consider equations of the type

$$\frac{du}{d\lambda} = \varepsilon \sum_{k=1}^n f_k(u) r_k(\lambda) \equiv \varepsilon H(\lambda, u), \quad (1)$$

$$u(\lambda_0) = u_0 \quad (2)$$

on the interval

$$\lambda_0 \leq \lambda \leq \Lambda = \lambda_0 + O(1/\varepsilon), \quad (3)$$

where $u = \{u^1, \dots, u^m\}$ and $f_k = \{f_k^1, \dots, f_k^m\}$. The $f_k(u)$ are given on a certain set U of an m -dimensional real Euclidean space, and ε is a small parameter. The distinctive feature of the problem is the large interval $[\lambda_0, \Lambda]$, on which the integration of equation (1) must be performed.

We assume that the functions $r_k(\lambda)$ are bounded on $[\lambda_0, \infty)$, and that the functions $f_k(u)$ are bounded on U together with several of their first derivatives.

Let us suppose that the approximate solution of problem (1), (2) is obtained in the form of the table

$$\lambda_0, \lambda_1, \dots, \lambda_N = \Lambda, \quad (4)$$

$$u_0, u_1, \dots, u_N. \quad (5)$$

Denote by $v_i(\lambda)$ the exact solution of equation (1) with the initial condition $v_i(\lambda_i) = u_i$, and by Δ_i the integration error on one step:

$$\Delta_i = v_i(\lambda_{i+1}) - u_{i+1}.$$

We study methods of numerical integration for problem (1), (2) under the following assumptions:

A. The exact solution $u(\lambda)$ of problem (1), (2) exists and is unique, and $u(\lambda) \in U$ ($\lambda_0 \leq \lambda \leq \Lambda$).

B. The points u_i belong to U ($i = 0, \dots, N$).

C. The methods under consideration are stable in the sense that there exists a constant K , depending on the data of the problem and on the chosen method, but not depending on the table (4), such that

$$|u(\lambda_j) - u_j| \leq K \sum_{i=0}^{N-1} |\Delta_i| \quad (j = 1, \dots, N). \quad (6)$$

We devote a few lines to explaining why, for problem (1), (2), the usual methods of numerical integration turn out to be of little use. For definiteness, let us consider the method of expansion by Taylor's formula. In this method, if λ_k, u_k have been obtained, then equation (1) and the equations obtained from it by successive differentiation are used to compute several derivatives of the functions $v_i(\lambda)$ at the point λ_i , after which

assume, for some p ,

$$u_{i+1} = \sum_{k=0}^p \frac{d^k v_i(\lambda_i)}{d\lambda^k} \frac{1}{k!} h^k \quad (h = \lambda_{i+1} - \lambda_i). \quad (7)$$

These calculations can be arranged as follows. For any sufficiently smooth function $G(\lambda, u)$, introduce the transformation

$$\begin{aligned} G(\lambda, v_i(\lambda)) &= G(\lambda_i, u_i) + \int_{\lambda_i}^{\lambda} \frac{dG(\nu, v_i(\nu))}{d\nu} d\nu = \\ &= G(\lambda_i, u_i) + \int_{\lambda_i}^{\lambda} \left[\frac{\partial G(\nu, v_i(\nu))}{\partial \nu} + \frac{\partial G(\nu, v_i(\nu))}{\partial u} \varepsilon H(\nu, v_i(\nu)) \right] d\nu. \end{aligned} \quad (8)$$

Owing to the presence of the term $\partial G/\partial \nu$, the second term on the right-hand side of (8) will be of order $\lambda - \lambda_i$. Application of the transformation (8) to the function H and to its total derivatives with respect to λ leads to formula (7), supplemented by the remainder term

$$\Delta_i = \varepsilon \int_{\lambda_i}^{\lambda_{i+1}} d\nu_1 \int_{\lambda_i}^{\nu_1} d\nu_2 \dots \int_{\lambda_i}^{\nu_p} \frac{d^p H(\nu_{p+1}, v_i(\nu_{p+1}))}{d\nu_{p+1}^p} d\nu_{p+1}.$$

The quantity Δ_i is of order εh^{p+1} , where $h = \lambda_{i+1} - \lambda_i$. The right-hand side in formula (6) will have order h^p , and it will be small only when $h \ll 1$. (Here, for simplicity, the step h is regarded as independent of i .) In this case the number

N has to be taken considerably larger than $1/\varepsilon$, which is associated with very laborious computations.

We propose a method close to the method that uses Taylor's formula. However, instead of the transformation (8), we use the analogous transformation of a function depending only on u :

$$F(v_i(\lambda)) = F(u_i) + \varepsilon \int_{\lambda_i}^{\lambda} \frac{\partial F(v_i(\nu))}{\partial v_i} H(\nu, v_i(\nu)) d\nu. \quad (9)$$

The second term here is of order $\varepsilon(\lambda - \lambda_i)$.

Let $\varphi_k(\lambda)$ ($k = 1, 2, \dots$) be functions, each of which coincides with one of the functions $r_k(\lambda)$ ($k = 1, \dots, m$), and let, for an arbitrary function $F(u)$, the notation

$$\begin{aligned} S_q(\varphi_1, \varphi_2, \dots, \varphi_q; F) &= \int_{\lambda_i}^{\lambda_{i+1}} \varphi_1(\nu_1) d\nu_1 \int_{\lambda_i}^{\nu_1} \varphi_2(\nu_2) d\nu_2 \dots \\ &\dots \int_{\lambda_i}^{\nu_{q-1}} \varphi_q(\nu_q) F(v_i(\nu_q)) d\nu_q. \end{aligned} \quad (10)$$

Applying the transformation (9), we obtain

$$\begin{aligned} S_q(\varphi_1, \dots, \varphi_q; F) &= F(u_i) S_q(\varphi_1, \dots, \varphi_q; 1) + \\ &+ \varepsilon \sum_{k=1}^n S_{q+1} \left(\varphi_1, \dots, \varphi_q, r_k; \frac{\partial F}{\partial u} f_k \right). \end{aligned} \quad (11)$$

Here, generally speaking, the left-hand side and the first term on the right have order $(\lambda - \lambda_i)^q$, while the second term on the right has order $\varepsilon(\lambda - \lambda_i)^{q+1}$. Thus, the second term will be small in comparison with the first only if $\varepsilon(\lambda - \lambda_i)$ is a small quantity, which is also possible when the difference $\lambda - \lambda_i$ is large in comparison with unity.

In connection with this, let us take $h = \lambda_{i+1} - \lambda_i$ (generally speaking, h depends on i) to be of order $1/\sqrt{\varepsilon}$, and carry out the transformations:

$$\begin{aligned} v_i(\lambda_{i+1}) - u_i &= \varepsilon \sum_{k_1=1}^n S_1(r_{k_1}; f_{k_1}) = \\ &= \varepsilon \sum_{k_1=1}^n f_{k_1}(u_i) S_1(r_{k_1}; 1) + \varepsilon^2 \sum_{k_1, k_2=1}^n S_2 \left(r_{k_1}, r_{k_2}; \frac{\partial f_{k_1}}{\partial u} f_{k_2} \right) = \end{aligned}$$

$$\begin{aligned}
 &= \varepsilon \sum_{k_1=1}^n f_{k_1}(u_i) S_1(r_{k_1}; 1) + \varepsilon^2 \sum_{k_1, k_2=1}^n \frac{\partial f_{k_1}(u_i)}{\partial u} f_{k_2}(u_i) S_2(r_{k_1}, r_{k_2}; 1) + \\
 &\quad + \varepsilon^3 \sum_{k_1, k_2, k_3=1}^n S_3 \left(r_{k_1}, r_{k_2}, r_{k_3}; \frac{\partial}{\partial u} \left(\frac{\partial f_{k_1}}{\partial u} f_{k_2} \right) f_{k_3} \right) = \dots
 \end{aligned}$$

We have found a representation of the difference $v_i(\lambda_{i+1}) - u_i$ in the form of a sum whose terms have orders $\varepsilon^{1/2}, \varepsilon, \varepsilon^{3/2}, \dots$. Taking

$$\begin{aligned}
 u_{i+1} &= u_i + \varepsilon \sum_{k_1=1}^n f_{k_1}(u_i) S_1(r_{k_1}; 1) + \dots \\
 &\quad \dots + \varepsilon^p \sum_{k_1, \dots, k_p=1}^n S_p(r_{k_1}, \dots, r_{k_p}; 1) \frac{\partial}{\partial u} (\dots f_{k_{p-1}}(u_i)) f_{k_p}(u_i), \quad (12)
 \end{aligned}$$

we obtain

$$\Delta_i = \varepsilon^{p+1} \sum_{k_1, \dots, k_{p+1}} S_{p+1} \left(r_{k_1}, \dots, r_{k_{p+1}}; \frac{\partial}{\partial u} (\dots f_{k_p}) f_{k_{p+1}} \right) = O(\varepsilon^{(p+1)/2}).$$

Computationally, the matter reduces to finding the multiple integrals $S_1(r_{k_1}; 1), \dots, S_p(r_{k_1}, \dots, r_{k_p}; 1)$, which can sometimes be done by means of analytic calculations. In general, however, one has to resort to mechanical quadratures. The number of steps N will be a quantity of order $1/\sqrt{\varepsilon}$, and the right-hand side of formula (6) will have order $\varepsilon^{p/2}$.

Remark. The assertion that the difference between the exact and approximate solution is a quantity of order $\varepsilon^{p/2}$ applies only to the arguments contained in table (4). In the intervals between the nodes, the difference between $u(\lambda)$ and the values obtained from table (5) by interpolation may be considerably larger. Thus, if $r_k(\lambda)$ are trigonometric functions of multiple angles, this difference will be of order ε .

Let us also note in conclusion that the method set forth, without substantial changes, can be applied to a system of the form $du/d\lambda = \varepsilon g(u, \varepsilon\lambda, \lambda)$, where g is a sufficiently smooth function of the first two arguments.

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Note: Figure translations are in progress. See original paper for figures.

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