ON THE CARLEMANNESS OF THE RESOLVENTS OF CERTAIN LINEAR OPERATORS

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ON THE CARLEMANNESS OF THE RESOLVENTS OF CERTAIN LINEAR OPERATORS

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In the theory of expansions in generalized eigenfunctions of self-adjoint operators, the Carleman property of the resolvents* of these operators is used in an essential way ([1−3]).

The present article is devoted to finding necessary and sufficient conditions under which the resolvent of a linear unbounded operator is an integral operator of Carleman type (for the definition of a Carleman operator see ([4]), p. 748).

Theorem 1. *Let Ω be a certain domain in the Euclidean space R_n. Let T be a linear operator defined on an everywhere dense linear manifold D_T in $L_2(\Omega)$, and let λ be a regular point** of the operator T. In order that the operator $R_\lambda = (T - \lambda E)^{-1}$ be an integral operator of Carleman type, it is necessary and sufficient that there exist a partition of the domain Ω into a finite or countable set of nonintersecting bounded measurable sets Ω_m, $m = 1, 2, ...$, such that for every function $f \in D_T$, $f \in L_\infty(\Omega_m)$, $m = 1, 2, ...$*

Proof. Necessity. Let $R_\lambda = (T - \lambda E)^{-1}$ be an integral operator of Carleman type, i.e.

$$(R_\lambda f)(s) = \int_\Omega K(s,t) f(t)\, dt, \qquad f \in L_2(\Omega),$$

where $K(s,t)$ is a measurable complex-valued function defined on $\Omega \times \Omega$, satisfying the condition

$$\int_\Omega |K(s,t)|^2\, dt < \infty \qquad \text{for almost all } s \in \Omega.$$

Consider the function

$$K(s) = \left(\int_\Omega |K(s,t)|^2\, dt \right)^{1/2}.$$

$K(s)$ is a measurable, finite almost everywhere function. Let

$$\Omega_m = \left(s = (s_1, \dots, s_n),\ m-1 < K(s) \le m,\ \max_{j=1,2,\dots,n} |s_j| \le m \right),$$

$m = 1, 2, \dots$, be measurable, nonintersecting bounded sets. Let $f \in D_T$. Then there exists a function $g \in L_2(\Omega)$ such that $f = R_\lambda g$. Moreover, for almost all $s \in \Omega_m$, $m = 1, 2, \dots$,

$$|f(s)| = |(R_\lambda g)(s)| = \left| \int_\Omega K(s,t) g(t)\, dt \right| \le$$

$$\le \left(\int_\Omega |K(s,t)|^2\, dt \right)^{1/2} \|g\| = K(s)\|g\| \le m\|g\|.$$

* Or of some power of the resolvent.
** That is, the operator $(T - \lambda E)^{-1}$ exists, is defined on the whole space $L_2(\Omega)$, and is bounded.

Sufficiency. Let $\Omega_m,\ m = 1, 2, \dots$, be the partition of the domain Ω indicated in the condition of Theorem 1. Define the operator P_m by the equality

$$P_m f = \chi_{\Omega_m}(s) f(s), \qquad f \in L_2(\Omega),$$

where χ_{Ω_m} is the characteristic function of the set Ω_m. The operator P_{mR_λ} is a restricted operator acting in $L_2(\Omega)$. At the same time, the range of the operator P_{mR_λ} belongs to $L_\infty(\Omega)$. It is not difficult to verify that the operator $P_{mR_\lambda},\ m = 1, 2, \dots$, is a closed operator acting from $L_2(\Omega)$ into $L_\infty(\Omega)$. Hence it follows ((5), p. 70) that P_{mR_λ} is a bounded operator acting from $L_2(\Omega)$ into $L_\infty(\Omega)$.

Thus,

$$\operatorname*{vrai\,sup}_{x \in \Omega_m} |(R_\lambda g)(s)| \le C_m \|g\|, \qquad m = 1, 2, \dots. \tag{1}$$

Define the function $\Lambda(s)$ by the equality $\Lambda(s) = C_m$ if $s \in \Omega_m,\ m = 1, 2, \dots$. The function $\Lambda(s)$ is measurable and finite almost everywhere. From (1) and the definition of the function $\Lambda(s)$ it follows that, for almost all $s \in \Omega$,

$$|(R_\lambda g)(s)| \le \Lambda(s)\|g\|, \qquad g \in L_2(\Omega).$$

Hence, by virtue of (4), it follows that R_λ is an integral operator of Carleman type.

Corollary 1. *Let λ be a regular point of a linear manifold D_T, dense everywhere in $L_2(\Omega)$, of the linear operator T. Suppose that for every function $f \in D_T$ and* *every compact subset $\widetilde{\Omega} \subset \Omega$ one has $f\chi_{\widetilde{\Omega}} \in L_\infty(\Omega)$. Then $(T-\lambda E)^{-1}$ is an integral operator of Carleman type.*

Proof. Cover the domain Ω by a system of pairwise disjoint half-open cubes d_k lying strictly inside Ω. Consider the closure $\overline{d}_k$ of the cube d_k. Since, by the assumption, $f\chi_{\overline{d}k} \in L_\infty(\Omega)$, $f \in D_T$, it follows by Theorem 1 that $(T-\lambda E)^{-1}$ is an integral operator of Carleman type.

Theorem 2. *Let λ be a regular point of the operator T, $\overline{D}_T = L_2(\Omega)$. In order that the operator $(T-\lambda E)^{-1}$ be an integral operator of Carleman type, it is necessary and sufficient that, for every $\varepsilon > 0$, there exist a measurable set E_ε, $mE_\varepsilon < \varepsilon$, such that for every function $f \in D_T$ one has $f\chi_{\Omega \setminus E_\varepsilon} \in L_\infty(\Omega)$.*

Proof. Necessity. Let $f \in D_T$. Then there exists $g \in L_2(\Omega)$ such that

$$f = (T-\lambda E)^{-1} g = \int_\Omega K(s,t) g(t)\, dt.$$

Consider the function

$$K(s) = \left(\int_\Omega |K(s,t)|^2\, dt \right)^{1/2}.$$

Since $K(s)$ is a measurable finite function almost everywhere, for any $\varepsilon > 0$ there exists a set E_ε, $mE_\varepsilon < \varepsilon$, such that for all $s \in \Omega \setminus E_\varepsilon$ one has $K(s) \le K$. But then, for almost all $s \in \Omega \setminus E_\varepsilon$,

$$|f(s)| = |(T-\lambda E)^{-1} g(s)| = \left| \int_\Omega K(s,t) g(t)\, dt \right| \le K(s)\|g\| \le K\|g\|.$$

Sufficiency. From the condition of Theorem 2 it follows that there exists a sequence of measurable sets E_m, $E_1 \subset E_2 \subset \dots \subset E_m \subset \dots \subset \Omega$, such that $f\chi_{E_m} \in L_\infty(\Omega)$, $f \in D_T$. Let

$$\Omega_{1,m} = E(s : s = (s_1, \dots, s_n),\ s \in E_1,\ m-1 < \max_{j=1,\dots,n} |s_j| \le m).$$

Set $G_l = E_l \setminus E_{l-1}$, $l = 2, 3, \dots$, and

$$\Omega_{l,j} = E(s : s = (s_1, \dots, s_n),\ s \in G_l,\ j-1 < \max_{i=1,\dots,n} |s_i| \le j).$$

The sets $\Omega_{l,j}$; $l = 1, 2, \dots$, $j = 1, 2, \dots$, are pairwise disjoint, measurable, bounded, and

$$\Omega = \bigcup_{j=1,\ l=1}^{\infty} \Omega_{l,j}.$$

At the same time $f\chi_{\Omega_{l,j}} \in L_\infty(\Omega)$, $f \in D_T$. By virtue of Theorem 1, it follows from this that $(T-\lambda E)^{-1}$ is an integral operator of Carleman type.

Theorem 3. Let λ be a regular point of the operator T, $\overline{D_T} = L_2(\Omega)$. In order that the operator $(T-\lambda E)^{-1}$ be an integral operator of Carleman type, it is necessary and sufficient that for every $\varepsilon > 0$ there exist a set E_ε, $mE_\varepsilon < \varepsilon$, such that, whatever function $f \in D_T$ is taken, the restriction of this function to the set $\Omega \setminus E_\varepsilon$ is a continuous bounded function after alteration on a set of measure zero.

Proof. Necessity. Let $(T-\lambda E)^{-1}$ be an integral operator of Carleman type, and let $K(s,t)$ be its kernel. As was shown in ([6]), the kernel $K(s,t)$ determines, by the equality $\overline{K(s,\cdot)} = \varphi(s)$, a measurable abstract function $\varphi(s)$ taking values in $L_2(\Omega)$. By virtue of the known generalization of N. N. Luzin' s theorem to the case of abstract functions, for every $\varepsilon > 0$ there is a set H_ε, $mH_\varepsilon < \varepsilon/2$, such that the restriction of the function $\varphi(s)$ to the set $\Omega \setminus H_\varepsilon$ will be a continuous abstract function. Since the function

$$K(s) = \|\varphi(s)\| = \left(\int_\Omega |K(s,t)|^2 \, dt\right)^{1/2}$$

is measurable and finite almost everywhere, for every $\varepsilon > 0$ there is a set $G_\varepsilon \subset \Omega \setminus H_\varepsilon$, $mG_\varepsilon < \varepsilon/2$, such that the restriction of the function $K(s) = \|\varphi(s)\|$ to the set $\Omega \setminus H_\varepsilon \cup G_\varepsilon$ will be a bounded function. Let $f \in D_T$. Then there is a $g \in L_2(\Omega)$ such that, after alteration on a set of measure zero, for all $s \in \Omega \setminus H_\varepsilon \cup G_\varepsilon$,

$$f(s) = ((T-\lambda E)^{-1} g)(s) = (g, \varphi(s)).$$

This implies the validity of the assertion being proved.

Sufficiency follows in an obvious way from Theorem 2.

Example 1. Let P be a maximal complete hypoelliptic operator with constant coefficients. Let λ be a regular point of P. Then there exists a natural number k such that $(P-\lambda E)^{-k}$ is an integral operator of Carleman type.

Proof. As is known (([7]), $p.105$), there is a natural number k such that a function $f \in D_{P^k}$, after alteration on a set of measure zero, will be a continuous function. We note that

$$D_{P^k} = D_{(P-\lambda E)^k},$$

and use Corollary 1 for the operator $(P-\lambda E)^{-k}$.

Example 2. Let $P(x,D)$ be a linear differential operator of constant strength with coefficients from $C^\infty(\Omega)$. Let the operator $P_0(D) = P(x_0, D)$ be hypoelliptic at some point $x_0 \in \Omega$. Let zero be a regular point of the operator $P(x,D)$. Then there is a natural number k such that $[P(x,D)]^{-k}$ is an integral operator of Carleman type.

This assertion follows from ([8]), 7.4.1, 2.3.7, and Corollary 1.

Example 3. Let L be a strongly elliptic operator of order $2m$ with coefficients from $C^\infty(\Omega)$, and let 0 be a regular point of L. Then L^{-k} is an integral operator of Carleman type if $4mk > n$.

By K. Friedrichs' theorem (([9]), p.248), L^{-k} acts from $L_2(\Omega)$ into $W_2^{2mk}(\Omega)$. If $4mk > n$, then by the imbedding theorem of S. L. Sobolev (([10]), p.64), $W_2^{2mk}(\Omega)$ is imbedded in $C(\Omega)$. By virtue of Corollary 1 it follows from this that L^{-k} is an integral operator of Carleman type.

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Note: Figure translations are in progress. See original paper for figures.