A CENTRAL LIMIT THEOREM FOR Y-FLOWS ON THREE-DIMENSIONAL MANIFOLDS

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Abstract

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A CENTRAL LIMIT THEOREM FOR Y-FLOWS ON THREE-DIMENSIONAL MANIFOLDS

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1. In paper (1) a central limit theorem was proved for a broad class of functions in the case of a geodesic flow in the space of line elements of a compact manifold $\mathcal{L}$ of constant negative curvature. In the study of this class in (1), the properties of $\mathcal{L}$ as a homogeneous space and the representation of the group of its motions are essentially used. Therefore these methods are not applicable to the case of variable curvature.

In the present paper another method is indicated for proving the central limit theorem for Y-flows on three-dimensional manifolds.

2. Definition 1. A measurable essentially bounded real-valued function f, defined on a Lebesgue space M with measure m (2), in which a measurable ergodic flow $\{S^t\}$ acts, satisfies the central limit theorem if, for any fixed a, $-\infty < a < \infty$,

$$\lim_{t\to\infty} m\left\{x : \left(\int_0^t f(S^\tau x)\,d\tau - t\bar{f}\right) \Big/ \sqrt{D_t f} < a\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^a e^{-u^2/2}\,du,$$

where

$$\bar{f} = \int_M f(x)\,dm; \qquad D_t f = \int_M \left[\int_0^t (f(S^\tau x) - \bar{f})\,d\tau\right]^2 dm.$$

Consider a stationary random process (3) $\{x_n,\ n\text{—integer},\ -\infty < n < \infty\}$ with a finite number of states and with space (X, P) of its realizations. Let ξ_a^b be the partition of the space X generated by the random variables x_i for all $i:\ a \le i \le b$, and let $\mathfrak{M}_a^b$ be the σ-algebra generated by all measurable sets consisting mod 0 of elements of the partition ξ_a^b.

Definition 2. The process $\{x_n\}$ has **strong mixing** ([4,5]) if, for any $B_i \in \mathfrak{M}_n^\infty$, $B_i \cap B_j = \emptyset$, $A \in \mathfrak{M}_{-\infty}^0$,

$$\sum_i |P(B_i/A) - P(B_i)| \leq \varphi(n)$$

and $\varphi(n)$ tends to 0 as $n \to \infty$.

3. Let W be a three-dimensional compact Riemannian manifold of class C^∞, let $\{T^t\}$ be a Y-flow of class C^2 on W ([6]), all leaves of whose transversal foliations are everywhere dense in W, and let μ be the normalized measure in W, invariant with respect to $\{T^t\}$, constructed in ([7]). In ([7]) it is shown that the flow $\{T^t\} = (U, l)$ in (W, μ) is represented as a special flow ([2]), constructed from an automorphism U acting in (M, m), and a function l, where U has in M the properties of a Y-diffeomorphism, and l satisfies a Hölder condition of positive order.

Let S be an arbitrary Y-diffeomorphism of class C^2, acting in the space Y and having everywhere dense leaves of transversal fol-

and ν is an S-invariant measure constructed in (8). By the methods of (8) one can prove the following theorem.

Theorem 1. The Y-diffeomorphism S in (Y, ν) is isomorphic to the shift $\widetilde{S}$ in the space (X, P) of realizations of a stationary random process $\{x_n\}$ possessing strong mixing with function $\varphi(n) = C\lambda^{\sqrt{n}}$, where $0 < \lambda < 1$ and $C > 0$ is a constant.

4. **Theorem 2.** Let a continuous function F on (Y, ν) satisfy a Hölder condition of positive order and let $D_N(F) \sim aN$ as $N \to \infty$, where

$$D_N(F) = \int_Y \left[\sum_{k=1}^{N} (F(S^k y) - \bar{F})\right]^2 d\nu$$

and $a > 0$. Then F obeys the central limit theorem.

By Theorem 1, F is a function on the space of realizations (X, P) of the stationary process $\{x_n\}$. Let C_{-k}^k be the element of the partition ξ_{-k}^k containing x. Put

$$F_k(x) = \int_{C_{-k}^k(x)} F(u)\, dP_{C_{-k}^k},$$

where integration is with respect to the conditional measure induced by the measure P on C_{-k}^k. From the properties of Y it follows that the norm in $\mathcal{L}_P^2(X)$ is

$$\|F - F_k\| < A\gamma^{\alpha k}, \tag{*}$$

where $0 < \gamma < 1$ and $A > 0$ is a constant.

Therefore Theorem 2 follows from Theorem 1 and Theorem 5 in (4). Theorems 1 and 2 are also valid for the automorphism U in (M, m).

Let $\widehat{S}$ be the unitary operator in $\mathcal{L}^2_P(X)$ conjugate to $\widetilde{S}$. Every function $g \in \mathcal{L}^2_P(X)$ has a Lebesgue spectrum with respect to $\widehat{S}$. Let $r_g(\rho)$ be the spectral density of g. In (9) it is shown that if: 1) $r_g(\rho)$ is continuous at the point $\rho = 0$ and 2) $r_g(0) = r_0 > 0$, then $D_N(g) \sim aN$ and $a = 2\pi r_0$. If g is continuous with a Hölder condition, then from Theorem 1 and relation $(*)$ it is seen that its correlation function decreases exponentially. In this case (9) ensures the fulfillment of requirements 1-2 when the equation

$$\widehat{S}\varphi - \varphi = g - \bar{g} \qquad \left(\bar{g} = \int_X g(x)\, dP\right)$$

has no solution in $\mathcal{L}^2_P(X)$.

It can be shown that if the transversal foliations of our Y-flow form a nonintegrable pair (6), then for a function l of the special representation the equation

$$\widehat{U}\varphi - \varphi = l - \bar{l}$$

has no solution in $\mathcal{L}^2_m(M)$, i.e. Theorem 2 is valid for l.

5. Let V be the infinitesimal operator corresponding to the group $\{V^t\}$ of unitary operators conjugate to the flow $\{T^t\}$. Let $f \in \mathcal{L}^2_\mu(W)$. Put, for $x \in M$,

$$F(x) = \int_0^{l(x)} f(T^t x)\, dt; \qquad H(x) = F(x) - \bar{f}\, l(x).$$

Theorem 3. Let a continuous function f on (W, μ) satisfy a Hölder condition of positive order, and suppose that the equation $Vh = f - \bar{f}$ has no solution in $\mathcal{L}^2_\mu(W)$. Then f obeys the central limit theorem; moreover, in the relation $D_T f \sim \sigma T$ the quantity

$$\sigma = \frac{2\pi}{\bar{l}}\, r_H(0).$$

The proof of Theorem 3 uses the special representation and Theorems 1 and 2. Note that if the equation $Vh = f - \bar{f}$ has no solution in $\mathcal{L}^2_\mu(W)$, then the equation $\widehat{U}g - g = H$ has no solution in $\mathcal{L}^2_m(M)$, and then Theorem 2 is valid for

H. Theorem 3 is proved by verifying that the limiting (as $t \to \infty$) distribution of the normalized integral-

of

$$\int_0^t f(T^u w)\,du$$

in (W, μ) coincides with the limiting (as $N \to \infty$) distribution of the normalized sum

$$\sum_{k=0}^{N} H(U^k x)$$

in (M, m).

Theorem 3 is valid for characteristic functions of sets with sufficiently smooth boundary.

For the geodesic flow on a compact manifold of negative curvature, the measure μ coincides with the invariant Riemannian volume. The class of functions in Theorem 3 coincides with the class of functions in (1).

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Note: Figure translations are in progress. See original paper for figures.