ON CONTINUOUS PARTITIONS OF COLLECTIVELY NORMAL SPACES

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Abstract

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ON CONTINUOUS PARTITIONS OF COLLECTIVELY NORMAL SPACES

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Definition 1. Let there be a set L of cardinality τ, and let to each element λ of L there be assigned a discrete system of sets γ_λ. Let

$$\gamma = \bigcup_{\lambda \in L} \gamma_\lambda.$$

We shall say that the system γ is τ-discrete.

Definition 2. Let X be a completely regular space. We shall say that its **uniform weight is equal** to τ if there exists a uniform structure with a base of τ covers, compatible with the topology of the space X, and for any cardinal number $\sigma < \tau$ no such uniform structure exists.

Theorem. *Let X be a collectively normal space of uniform weight τ, and let $\{A\}$ be its continuous partition. Then in all elements of the partition, except for the elements of some τ-discrete system, every discrete set has cardinality $< \tau$.*

Proof. 1. Denote by $\{\alpha_\lambda\}_{\lambda \in L}$ a base of the uniform structure in X, consisting of τ covers α_λ. The system $\{\alpha_\lambda\}$ has the following properties: a) for any two covers α_{λ_1} and α_{λ_2} there exists a third, which is inscribed in their intersection $\alpha_{\lambda_3} < \alpha_{\lambda_1} \wedge \alpha_{\lambda_2}$; b) if x is a point in the space X, then the system of all stars of this point with respect to the covers α_λ forms a base at the point x.

2. To each λ from the set L we assign a discrete system of elements of the partition $\{A\}$. For this purpose denote by $V_\lambda(A)$ the star of the element A with respect to the cover α_λ. Into the system γ_λ we select those and only those elements of the partition which have the property: the star of the element of the partition with respect to the system α_λ contains wholly no element of the partition distinct from it itself,

$$A_0 \in \alpha_\lambda \Longleftrightarrow V_\lambda(A_0) \cap A_1 \neq A_1$$

for $A_1 \neq A_0$.

The discreteness of the system α_λ follows from the continuity of the partition $\{A\}$: if $A_0 \notin \gamma_\lambda$, then for any $x \in A_0$ there exists a neighborhood Ox which does not meet the bodies G_λ of the system α_λ. Otherwise one would have $[G_x] \cap A_0 \neq \Lambda$, and by virtue of the continuity of the partition the neighborhood $V_\lambda(A_0)$ of the set A_0 would contain wholly some element $A_1 \in \alpha_\lambda$, which contradicts the construction of the system α_λ.

3. We shall prove that a discrete subset of any element of the partition which has not entered into any system α_λ has cardinality $< \tau$. Suppose this is not so: $A_0 \notin \bigcup_{\lambda \in L} \gamma_\lambda$ and $A_0 \supset P$, where P is a discrete set of cardinality τ. Since L has cardinality τ, one can establish a one-to-one correspondence between P and L; then each point $x \in P$ acquires an index λ: $P = \{x_\lambda\}_{\lambda \in L}$.
4. Since X is collectively normal, the discrete system of points $\{x_\lambda\}$ has a discrete system of neighborhoods $\{Ox_\lambda\}$ in the space X.
5. For each point $x_\lambda \in P$ one can choose a cover $a(x_\lambda) \in \{\alpha_\lambda\}$ so that the star of this cover with respect to the point x_λ is contained in the set Ox. It may happen that the same cover α_λ corresponds to different points x_1 and x_2 of P. We have constructed a mapping φ of the set $P = \{x_\lambda\}$ into the set of covers $\{\alpha_\lambda\}_{\lambda \in L}$.
6. In item 3 a one-to-one correspondence between the sets L and P was established. It can be turned into a one-to-one mapping $\psi : \{x_\lambda\} \to \{\alpha_\lambda\}$, by assigning to each point x_λ the cover with the same index.
7. Let us construct a third mapping ω of the set P into the set of covers α_λ. Using property b) of item 1, to each point $x \in P$ we assign some cover α_x belonging to the system $\{\alpha_\lambda\}$, in such a way that, for any λ, one has $\alpha_x^\lambda < \alpha(x_\lambda) \cap \alpha_\lambda$, or $\omega(x) < \varphi(x) \cap \psi(x)$ $(x \in P)$.
8. For any $x \in P$ we have:

$$St(\alpha_x, x) \subset Ox; \tag{1}$$

there exists an element A_x of the decomposition $\{A\}$ such that

$$St(\alpha_x, A_x) \supset A. \tag{2}$$

9. Since $\{\alpha_x\}_{x \in P}$ forms a base of the uniform structure, we have $\left[\bigcup_{x \in P} A_x\right] \supset A$.
10. Consider an arbitrary A_x. In view of (2), there exists an element H of the cover α_x which intersects A_x and contains the point x. From (1) it follows that $A_x \cap Ox \neq \Lambda$. Choose in the set $A_x \cap Ox$ a point $y(x)$. Obviously, the set Q of all points $y(x)$ is discrete.

11. Consider the open set $X \setminus Q$. Since the decomposition $\{A\}$ is continuous, there exists an open saturated set $U \subset X \setminus Q$ which contains A_0. It is easy to see that $X \setminus Q$ does not contain entirely any one of the sets A_x $(x \in P)$; hence, for any $x \in P$, $A_x \cap U = \Lambda$ and $(\bigcup A_x) \cap U = \Lambda$, which contradicts the fact that $\left[\bigcup_{x \in P} A_x\right] \supset A_0$. The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.