# ON BASES OF THE SPACE OF CONTINUOUS FUNCTIONS

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## Abstract

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*MATHEMATICS*

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# ON BASES OF THE SPACE OF CONTINUOUS FUNCTIONS

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**1.** As early as J. Schauder (see ($^1$, $^2$), pp. 63-64) constructed a sequence of continuous functions $\{f_n(x)\}_{n=0}^{\infty}$ which is a basis of the space $C(0,1)$. The Schauder system has the form:

$$f_0(x) = 1 - x, \qquad f_1(x) = x,$$

$$f_n(x) = \begin{cases} 2^{p+1}x - 2q, & \text{for } q/2^p \leqslant x \leqslant (2q+1)/2^{p+1}, \\ 2q + 2 - 2^{p+1}x, & \text{for } (2q+1)/2^{p+1} < x \leqslant (q+1)/2^p, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = 2^p + q + 1,\ 0 \leqslant q \leqslant 2^p - 1,\ p = 0, 1, 2, \ldots.$

Let us pose the following problem. Suppose a certain function $\varphi(x)$, defined on $[0,1]$, is given. Under what conditions imposed on this function does the sequence

$$\varphi_0(x) = 1 - \varphi(x), \qquad \varphi_1(x) = \varphi(x),$$

$$\varphi_n(x) = \begin{cases} \varphi(2^{p+1}x - 2q), & \text{for } q/2^p \leqslant x \leqslant (q+1)/2^{p+1}, \\ \varphi(2q + 2 - 2^{p+1}x), & \text{for } (2q+1)/2^{p+1} < x \leqslant (q+1)/2^p, \\ 0 & \text{otherwise,} \end{cases}$$

where $n = 2^p + q + 1,\ 0 \leqslant q \leqslant 2^p - 1,\ p = 0, 1, 2, \ldots$, form a basis of the space $C(0,1)$?

The Schauder basis is obtained when $\varphi(x) = x$.

K. M. Shaidukov in ($^3$) proved that the system (1) forms a basis for $\varphi(x) = x(2-x)$ and $\varphi(x) = x^2(2-x)^2$.

Let

$$\Delta_h^{(2)}(f(t); x) = f(x) - 2f(x+h) + f(x+2h),$$
$$h > 0, \quad 0 \leqslant x < x + 2h \leqslant 1.$$

**Theorem 1.** *Let $\varphi(x)$ be a continuous nondecreasing function on the interval $[0,1]$ satisfying the following conditions:*

*a) for some nonnegative integer $s_0$ the equalities*

$$\varphi(k/2^{s_0}) = k/2^{s_0} \quad \text{for all } k = 0, 1, \dots, 2^{s_0};$$

*b) there exists a positive number $M < 3 \cdot 4^{s_0}$ such that for any $0 < h \leqslant 1/2$ and $x \in [0,1]$ the inequality*

$$\left|\Delta_h^{(2)}(\varphi(t); x)\right| \leqslant Mh^2$$

*holds.*

*Then the system of functions (1) forms a basis of the space of continuous functions.*

**2.** We now construct concrete bases that will be needed in what follows.

**Theorem 2.** Let the sequence $\{\Phi_m(x)\}_{m=1}^{\infty}$ be defined by the recurrence relation

$$\Phi_{m+1}(x) = \begin{cases} \displaystyle\int_0^{2x} \Phi_m(t)\,dt, & 0 \le x \le \frac{1}{2}, \\ \displaystyle -\int_0^{2-2x} \Phi_m(t)\,dt + 1, & \frac{1}{2} < x \le 1, \end{cases} \qquad m = 1, 2, \dots,$$

where $\Phi_1(x) = x$.

Then the sequence $\Phi_m(x)$ converges uniformly to the continuous function $\Phi_0(x)$.

**Theorem 3.** Let an integer $m \geq 0$ be fixed. Then the system of functions (1) with $\varphi(x) = \Phi_m(x)$ forms a basis of the space $C(0,1)$.

3. We now apply the results obtained to the problem of the order of growth of the degrees of a polynomial basis.

The problem in question is the following. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a polynomial basis of the space $C(0,1)$, and let $\nu_n$ be the degree of the polynomial $P_n(x)$. How small can the growth of $\nu_n$ be made?

This problem was first posed by Ch. Fejas and I. Singer in 1961 in the paper (4). This and the analogous problem on the order of growth of the degrees of an orthogonal polynomial basis was also independently posed by P. L. Ul' yanov in a survey lecture in the same year at the IV All-Union Mathematical Congress ((5), p. 698; see also (6), p. 18).

Ch. Fejas and I. Singer in (4) proved that there exists a polynomial basis with $\nu_n = 2^n$. Then K. M. Shaidukov (7) proved the theorem (see also (6), p. 19): for any sequence of natural numbers

$$\{i_n\}_{n=0}^{\infty}$$

satisfying the condition

$$\sum_{n=1}^{\infty} \frac{n}{i_n} < \infty,$$

one can construct a polynomial basis with $\nu_n \le i_n,\ n = 0, 1, 2, \ldots.$ In (8, 9) we strengthened all the preceding results.

On the other hand, in 1914 G. Faber (10) proved that it is impossible to construct a polynomial basis of the space $C(0, 1)$ with $\nu_n = n$. In this direction there are also assertions by V. F. Nikolaev, S. E. Lozinskii, and F. I. Kharshiladze (see (11, 12) and (13), p. 323).

In (14) we proved that the case $\nu_n = n + o(\ln n)$ is also impossible.

Relying on the results of Sections 1 and 2, one can prove that the following two theorems are valid.

Namely, if one takes the bases of Theorem 3 with $m > 0$, then by a method analogous to that of (8) one proves:

**Theorem 4.** Let a sequence of nonnegative integers $\{\mu_n\}_{n=0}^{\infty}$ for some natural number $m$ satisfy the conditions $\mu_n^m / n^{m-1} \uparrow \infty$ and

$$\sum_{n=1}^{\infty} \frac{n^{m-1}}{\mu_n^m} < \infty.$$

Then one can construct a polynomial algebraic basis of the space $C(0, 1)$ with $\nu_n \le \mu_n,\ n = 0, 1, 2, \ldots.$

In particular, for $m = 1$ Theorem 1 of (8) is obtained.

**Corollary.** For every $\varepsilon > 0$ there exists a polynomial basis $\{P_n(x)\}_{n=0}^{\infty}$ such that $\nu_n \le n \ln^{\varepsilon} n$ for $\varepsilon \ge 3$.

Relying on the basis of Theorem 3 with $m = 0$, we obtain:

**Theorem 5.** For any increasing sequence of nonnegative integers $\{\mu_n\}_{n=0}^{\infty}$ satisfying the condition

$$\sum_{n=1}^{\infty} \frac{\ln n}{n} \left(\frac{n}{\mu_n}\right)^{1/2} \ln \mu_n^{-1/2 \ln n} < \infty,$$

where the general term of this series tends monotonically to zero, one can construct a polynomial algebraic basis of the space $C(0, 1)$ with
$v_n \le 4\mu_n,\ n = 0, 1, 2, \ldots.$

**Corollary.** For every $\varepsilon > 0$ there exists a polynomial basis of the space of continuous functions such that
$v_n \le n(\ln n)^{(2+\varepsilon)/\sqrt{\ln \ln n}}$ for $n > 15$.

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