ON A CERTAIN CLASS OF SEMIGROUPS

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Abstract

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ON A CERTAIN CLASS OF SEMIGROUPS

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In the present article an attempt is made to construct a general theory of the class L of strongly continuous on $(0, \infty)$ semigroups $T(t)$ of bounded linear operators in a Banach space X, for which the functions $T(t)x$ $(x \in X)$ have a summable singularity at zero (the precise definition of the class L is given below). Conditions are clarified under which a semigroup $T(t)$ from L has a generator, and the properties of this generator are investigated. The main result consists of necessary and sufficient conditions under which a linear operator U is the generator of a semigroup of class L. The class L naturally decomposes into narrower classes E of semigroups $T(t)$, for which the functions $T(t)x$ $(x \in X)$ are not only summable but also belong to a certain ideal space of functions E (for example, L_p, Orlicz, Lorentz, etc.); the conditions found for membership of the semigroup $T(t)$ in the class E in particular cases turn into known theorems on generators (see [1–3], where a detailed bibliography is given).

1. Let $T(t)$ be a semigroup of bounded linear operators in a Banach space X, strongly continuous on $(0, \infty)$. Recall that for every such semigroup there exists the limit

$$\omega_0 = \lim_{t\to\infty} \frac{\ln \|T(t)\|}{t}, \tag{1}$$

called the **type** of the semigroup. The equality

$$A_0 x = \lim_{t\to 0} \frac{T(t)x - x}{t} \tag{2}$$

defines the **infinitesimal operator** A_0 of the semigroup; if the operator A_0 admits a closure, then one says that the semigroup has the **generator** $A = \overline{A}_0$.

We shall say that the semigroup $T(t)$ belongs to the **class** L if the following conditions are satisfied:

a) The subspace

$$E_0 = \bigcup_{0<t<\infty} R[T(t)] \tag{3}$$

($R(C)$ is the range of the operator C) is dense in X.

b) For every $x \in X$ the function $T(t)x$ is summable on every finite interval.

For every semigroup $T(t)$ of class L, the domain $D(A_0)$ of the infinitesimal operator A_0 is dense in X. Thus the subspace H_0 of elements $x \in X$ for which

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t T(s)x\,ds = x. \tag{4}$$

is also dense in X.

Let $T(t)$ be a semigroup of class L, and let ω_0 be its type. Then for $\operatorname{Re}\lambda > \omega_0$ a linear continuous operator is defined in X by

$$R(\lambda)x = \int_0^\infty e^{-\lambda t} T(t)x\,dt. \tag{5}$$

Lemma 1. The following equalities hold

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \qquad (\operatorname{Re}\lambda, \operatorname{Re}\mu > \omega_0); \tag{6}$$

$$(\lambda I - A_0)R(\lambda)x = x \qquad (x \in H_0,\ \operatorname{Re}\lambda > \omega_0); \tag{7}$$

$$R(\lambda)(\lambda I - A_0)x = x \qquad (x \in D(A_0),\ \operatorname{Re}\lambda > \omega_0). \tag{8}$$

It is natural to make the supposition that $R(\lambda)$ is the resolvent of the generating operator A of the semigroup $T(t)$. Indeed, the following is true.

Theorem 1. A semigroup $T(t)$ of class L has a generating operator if and only if $T(t)x = 0$ $(0 < t < \infty)$ implies $x = 0$. If $T(t)$ has generating operator A, then $R(\lambda, A) = R(\lambda)$ for $\operatorname{Re}\lambda > \omega_0$.

The property of a semigroup $T(t)$ of class L that "from $T(t)x = 0$ $(0 < t < \infty)$ it follows that $x = 0$" is equivalent to the fact that, for some λ_0 $(\operatorname{Re}\lambda_0 > \omega_0)$, from $R(\lambda_0)x = 0$ it follows that $x = 0$.

Theorem 2. Let $T(t)$ be a semigroup of class L with generating operator A. Then:

a) If $x \in D(A)$, then

$$T(t)x - x = \int_0^t T(s)Ax\,dx, \tag{9}$$

and, consequently, the function $T(t)x$ is continuous on $[0, \infty)$.

b) If

$$T(t)x - x = \int_0^t T(s)z\,ds, \tag{10}$$

then $x \in D(A)$ and $Ax = z$.

c) $D(A^2) \subseteq D(A_0)$.

It is of interest to determine when the infinitesimal operator A_0 of a semigroup $T(t)$ of class L is closed, i.e., when it coincides with the generating operator A of the semigroup.

Theorem 3. Let $T(t)$ be a semigroup of class L. Then $A_0 = A$ if and only if $X = H_0$.

2. We now pass to the question of what conditions a given operator U must satisfy in order to be the generating operator of a semigroup of class L. Naturally, it is necessary to consider only closed operators with domain dense in X.

Theorem 4. In order that a linear closed operator U with domain dense in X be the generating operator of a semigroup $T(t)$ of class L of type ω_0, it is necessary and sufficient that, for some ω ($\omega > \omega_0$), the following conditions hold:

a) $\|R(\lambda, U)\| \le M$ for $\operatorname{Re}\lambda > \omega$;

b) there exist a nonnegative function $\varphi(t, x)$ $(0 < t < \infty,\ x \in X)$, continuous in the aggregate of variables, and a nonnegative function $\varphi(t)$ $(0 < t < \infty)$, satisfying the conditions

$$\varphi(t_1 t_2) \le \varphi(t_1)\varphi(t_2), \qquad \lim_{t\to\infty} \frac{\ln \varphi(t)}{t} < \omega, \tag{11}$$

such that

$$\varphi(t, x) \leqslant \varphi(t)\|x\|, \qquad \int_0^\infty \varphi(t, x)e^{\omega t}\,dt < \infty \quad (x \in X), \tag{12}$$

$$\|R^{(n)}(\lambda, U)x\| \leqslant \int_0^\infty e^{-\lambda t} t^n \varphi(t, x)\,dt \quad (x \in X,\ \operatorname{Re}\lambda > \omega;\ n = 0, 1, \ldots). \tag{13}$$

Moreover,

$$\|T(t)x\| \leqslant \varphi(t,x) \quad (x \in X), \qquad \|T(t)\| \leqslant \varphi(t). \tag{14}$$

3. A semigroup of class L will be called a semigroup of class L_0 if the function $T(t)$ is also summable on every finite interval.

Theorem 5. In order that a linear closed operator U with domain dense in X be the generating operator of a semigroup $T(t)$ of class L_0 of type ω_0, it is necessary and sufficient that, for some ω ($\omega > \omega_0$) and for some continuous nonnegative function $\varphi(t)$ ($0 < t < \infty$) satisfying conditions (11), such that

$$\int_0^\infty \varphi(t) e^{\omega t}\, dt < \infty, \tag{15}$$

the inequalities

$$\|R^{(n)}(\lambda, U)\| \leqslant \int_0^\infty e^{-\lambda t} t^n \varphi(t)\, dt \quad (\lambda > \omega;\ n = 0, 1, \ldots). \tag{16}$$

hold. Moreover,

$$\|T(t)\| \leqslant \varphi(t). \tag{17}$$

Semigroups of class L_0 were considered, in another connection, by P. E. Sobolevskii (4).

4. Let E be an ideal space of functions $\xi(t)$ defined on $[0, \infty)$; suppose that E contains all bounded functions that vanish outside some interval $[0, a]$, and that each function from E is summable on any interval $[0, a]$. By E_ω (ω a real number) we denote the ideal space of functions $\xi(t)$ for which $e^{\omega t}\xi(t) \in E$, with norm

$$\|\xi\|_\omega = \|e^{\omega t}\xi(t)\|.$$

A semigroup $T(t)$ of class L (class L_0) will be called a semigroup of class E (class E_0) if $\|T(t)x\| \in E_\omega$ ($x \in X$) (if $\|T(t)\| \in E_\omega$) for some ω. In the case $E = L_1$, the class E (E_0) coincides with the whole class L (L_0). In the case $E = L_\infty$, the classes E and E_0 coincide with the class C_0 of semigroups bounded at zero and, consequently, strongly continuous on $[0, \infty)$.

Theorems 4 and 5 make it possible to formulate conditions under which a given linear operator is the generating operator of some semigroup of class E or class E_0.

Theorem 6. In order that a linear closed operator U with domain dense in X be the generating operator of a semigroup of class E (E_0), it is necessary and sufficient that conditions a) and b) of Theorem 4 (Theorem 5) be satisfied;

moreover, the function $\varphi(t,x)$ (the function $\varphi(t)$) can be chosen so that $\varphi(t,x) \in E_\omega$ for $x \in X$ ($\varphi(t) \in E_\omega$).

Let us consider, in particular, the case where E is the space C_α of all measurable functions on $[0,\infty)$ for which the norm

$$\|\xi(t)\| = \sup_{0<t<\infty} |t^\alpha \xi(t)|;$$

here $0 \le \alpha < 1$. From the principle of uniform boundedness it follows that the classes C_α and $(C_\alpha)_0$ coincide; in the case under consideration, as the function $\varphi(t)$ one may always take the function $\varphi(t) = Mt^{-\alpha}e^{\omega_1 t}$, where M and ω_1 ($\omega_1 < \omega$) are certain constants. For this function $\varphi(t)$, the condition of Theorem 5, in view of

$$\int_0^\infty e^{-\lambda t} t^n \varphi(t)\, dt = \frac{M\Gamma(n-\alpha+1)}{(\lambda-\omega_1)^{n-\alpha+1}},$$

takes the quite simple form:

$$\left\|R^{(n)}(\lambda,U)\right\| = M\Gamma(n-\alpha+1)/(\lambda+\omega)^{n-\alpha+1} \qquad (\lambda > \omega;\ n = 0,1,\ldots). \quad (18)$$

Thus we arrive at the following assertion: *a linear closed operator U with domain dense in X is the generating operator of a semigroup $T(t)$ of class C_α of type ω_0 if and only if, for some M and ω ($\omega > \omega_0$), the inequalities (18) hold.* For $\alpha = 0$ this assertion coincides with the assertion of the classical Hille–Phillips–Miyadera theorem; for $\alpha > 0$ a close assertion was proved by P. E. Sobolevskii.

In conclusion we note that already for semigroups of class C_α ($\alpha > 0$) the "positivity" inequality

$$\lim_{\lambda\to\infty} \|\lambda R(\lambda)\| < \infty,$$

generally speaking, does not hold.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.