THE EQUATION OF TRANSPORT OF A PASSIVE ADMIXTURE BY FILTRATION MOTION

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THE EQUATION OF TRANSPORT OF A PASSIVE ADMIXTURE BY FILTRATION MOTION

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To describe the macrocharacteristics of a filtration flow at any fixed point of a filtration medium, as in ([1]), one considers the statistics of the radii of the cross sections of the pore channels r_i, characterized by the probability distribution function $f(r)$, and the statistics of the orientations of these channels in space; for this purpose probability distribution functions of the polar and azimuthal angles φ_i and ψ_i, $f_1(\varphi)$ and $f_2(\psi)$, are introduced. To each pore channel passing through a fixed point of the filtration medium there corresponds a pair of angles φ_i and ψ_i, which determine the orientation in space of the axis of the pore channel under consideration in a rectangular coordinate system xzy chosen for each fixed point of the medium. The x-axis is directed along the mean filtration velocity of the flow $\langle u\rangle$; the angle φ is measured from the x-axis in the plane xz, and the angle ψ is measured from the x-axis in the plane xy. The variation of the angles φ and ψ is determined by the condition that the microflows of fluid in the individual pore channels cannot have a velocity component opposite to the mean filtration transport $\langle u\rangle$. Consequently, according to the condition, the individual streams in the pore channels passing through a fixed point of the medium are oriented in the half-space $x > 0$, and the variation of the angles φ and ψ is bounded by the limits $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$, $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$. The mean velocity of the filtration flow is determined by the formula

$$\langle u\rangle = mB^{-1}\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 r^2 u\cos\varphi\cos\psi\,dr\,d\varphi\,d\psi, \qquad B = \int_0^\infty f r^2\,dr, \tag{1}$$

where m is the porosity, equal to the ratio of the volume of the flow-through pores to the total volume of the porous medium (including the skeleton). In the averaging operation expressed by formula (1), the quantity m has the meaning of the probability that a fixed point of the medium lies inside a flow-through pore channel; u is the flow velocity of the fluid averaged over the cross section of a fixed pore channel.

By analogy with ([1]), we shall start from the elementary equation of transport of an admixture in a pore channel, written while neglecting molecular diffusion transport of the admixture along the channel,

$$-\pi r_i^2 \frac{\partial P_i}{\partial n} u_i = \pi r_i^2 \frac{\partial P_i}{\partial t} + 2\pi r_i q_i, \tag{2}$$

where $\partial/\partial n$ is the derivative taken along the length in the direction of the spatial orientation of the longitudinal axis of the channel under consideration; P_i is the mean concentration of the admixture in the cross section of a cylindrical pore channel of radius r_i; q_i is the amount of admixture depositing on the walls of the pore channel per unit time per unit area (or, conversely, emitted by the wall into the flowing stream; then q_i is taken with the opposite sign). The quantity $\partial P_i/\partial n$ in equation (2) is considered, in the probabilistic sense, as a possible realization of the concentration gradient of the admixture in

fixed point; therefore, in the coordinate system x, z, y the derivative $\partial/\partial n$ is expressed by the relations: $\cos\varphi\cos\psi\,\partial/\partial n = \partial/\partial x$, $\sin\varphi\,\partial/\partial n = \partial/\partial z$, $\cos\varphi\sin\psi\,\partial/\partial n = \partial/\partial y$. Multiplying equation (2) by $mB^{-1} f_1 f_2 \cos^2\varphi \cos^2\psi$ and integrating it with respect to r from 0 to ∞ and with respect to φ and ψ within the limits from $-\frac{1}{2}\pi$ to $\frac{1}{2}\pi$, we obtain

$$-\frac{\partial}{\partial x} Q_x = mF_x \frac{\partial P}{\partial t} + mB^{-1} \int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2\, 2rq \cos^2\varphi \cos^2\psi \, dr\, d\varphi\, d\psi,$$

$$Q_x = mB^{-1} \int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2\, r^2 up \cos\varphi \cos\psi \, dr\, d\varphi\, d\psi, \tag{3}$$

$$F_x = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\varphi \cos^2\psi\, f_1 f_2 \, d\varphi\, d\psi,$$

where Q_x is the mean filtration flux of the admixture along the mean flow, i.e., along the direction $x > 0$. The quantity P in equation (3) denotes the mean value of the admixture concentration at a fixed point of the filtration medium. The quantity P is calculated by averaging over the admixture concentrations in the cross sections of the jets that could intersect the point under consideration; for simplicity it is assumed that the quantities P_i do not depend on the angles φ_i and ψ_i. In the case of parallel pore channels, the quantity

$$A = mB^{-1} \int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2\, 2rq \, dr\, d\varphi\, d\psi$$

expresses the amount of admixture deposited in a unit volume of the porous medium. We shall assume that the quantity A continues to retain this meaning also in the case of pore channels arbitrarily oriented in space. Taking the preceding remarks into account, we rewrite equation (3) in the form

$$-\partial Q_x/\partial x = mF_x\,\partial P/\partial t + \Phi_{xA}, \tag{4}$$

where Φ_x is some mean value of the product $\cos^2\varphi\cos^2\psi$.

The mean admixture fluxes in the directions z and y (transverse to the mean flow), Q_z and Q_y, owe their origin to the lateral spreading of the admixture away from the direction of the mean flow as a result of the arbitrary spatial orientation of the pore channels. For definiteness, let $Q_z > 0$ and $Q_y > 0$. Then the filtration transport of the admixture along z may be regarded as the difference of two mean filtration fluxes directed to opposite sides, transported by filtration through the plane xy: one, Q_z^+, formed by jets having a velocity component in the direction $z > 0$, and the other, Q_z^-, formed by jets having a velocity component in the direction $z < 0$. In an analogous way, the quantity Q_y is composed of oppositely directed fluxes Q_y^+ and Q_y^-. Here and below, the plus index denotes quantities associated with jets having a velocity component in the direction $z > 0,\ y > 0$, and the minus index denotes quantities associated with jets oriented in the direction $z < 0,\ y < 0$,

$$Q_z^{+,-} = mB^{-1}P_z^{+,-}\int_0^\infty\int_0^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} ff_1f_2\,r^2u\sin\varphi\,dr\,d\varphi\,d\psi;$$

$$Q_y^{+,-} = mB^{-1}P_y^{+,-}\int_0^\infty\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi} ff_1f_2\,r^2u\cos\varphi\sin\psi\,dr\,d\varphi\,d\psi.$$

Thus, $Q_z = Q_z^+ - |Q_z^-|$ and $Q_y = Q_y^+ - |Q_y^-|$. At the same time, by assumption, there is no mean transport of mass along the directions z and y.

Therefore, the difference between the quantities $|Q_z^+|, |Q_y^+|$ and the quantities $|Q_z^-|, |Q_y^-|$ is due to the difference between the mean impurity concentrations in the streams of the class $z > 0, y > 0$, P_z^+, P_y^+, and the mean concentrations in the streams of the class $z < 0, y < 0$, P_z^-, P_y^-, where $P = P_z^+ + P_z^- = P_y^+ + P_y^-$. The conditions $Q_z > 0, Q_y > 0$ imply the relations $P_z^+ > P_z^-$, $P_y^+ > P_y^-$, which, in turn, imply the inequalities $q_z^+ > q_z^-$, $q_y^+ > q_y^-$, since an increased impurity concentration in one or another group of streams entails an increased amount of impurity deposition in this group of streams on the walls of the pore channels. However, unlike the picture of heat transfer (1), in all streams both of the class $z > 0, y > 0$ and of the class $z < 0, y < 0$ passing through a fixed point of the medium, the impurity concentration will decrease along these streams in accordance with equation (2). Consequently, the mean drops in impurity concentration at a fixed point, calculated separately over streams of different classes, must be directed toward one another; therefore the mean drop in impurity concentration in the directions of the axes z and y should be calculated as the difference of such drops, calculated separately over the streams of the class $z > 0, y > 0$ and of the class $z < 0, y < 0$, i.e.

$$|\partial P/\partial z| = |\partial P^+/\partial z| - |\partial P^-/\partial z| \quad (\text{for } \partial P/\partial z < 0); \tag{5}$$

$$|\partial P/\partial y| = |\partial P^+/\partial y| - |\partial P^-/\partial y| \quad (\text{for } \partial P/\partial y < 0). \tag{6}$$

The decrease of the mean filtration flux of impurity along the axes z and y (i.e. across the mean flow of the liquid) can be obtained by averaging the projection of equation (2) onto the axes z and y. To do this, multiplying equation (2) first by $mB^{-1}f_1 f_2 \sin^2\varphi$, and then by $mB^{-1}f_1 f_2 \cos^2\varphi \sin^2\psi$, we integrate it over the limits $0 < r < \infty$; $-\frac{1}{2}\pi < \varphi < \frac{1}{2}\pi$; $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$. As a result we obtain

$$-\frac{\partial Q_z}{\partial z} = mF_z\frac{\partial P}{\partial t} + 2mB^{-1}\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 \sin^2\varphi\, q\, dr\, d\varphi\, d\psi; \tag{7}$$

$$-\frac{\partial Q_y}{\partial y} = mF_y\frac{\partial P}{\partial t} + 2mB^{-1}\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 \cos^2\varphi \sin^2\psi\, q\, dr\, d\varphi\, d\psi; \tag{8}$$

$$F_z = F_z^+ + F_z^-, \qquad F_z^{+,-} = \left| \int_0^{\frac{1}{2}\pi} f_1 \sin^2\varphi\, d\varphi \right|;$$

$$F_n = F_y^+ + F_y^-, \qquad F_y^{+,-} = \left| \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} f_1 f_2 \cos^2\varphi \sin^2\psi\, d\varphi\, d\psi \right|.$$

Adding the right- and left-hand sides of equations (4), (7), (8), we obtain an equation expressing the general balance of inflow and outflow of impurity in an elementary volume of a porous medium, i.e. an analogue of the classical diffusion equation,

$$-\partial Q_x/\partial x - \partial Q_z/\partial z - \partial Q_y/\partial y = m\,\partial P/\partial t + A. \tag{9}$$

We now write equations of the type (7) and (8), describing the transfer of impurity along the axes z and y separately for streams of the classes $z > 0, y > 0$ and $z < 0, y < 0$:

$$
\begin{aligned}
\alpha_z^+ \langle u \rangle \left| \partial P_z^+ / \partial z \right| &= m F_z^+ \partial P_z^+ / \partial t + \Phi_z^+ A_z^+; \\
\alpha_z^- \langle u \rangle \left| \partial P_z^- / \partial z \right| &= m F_z^- \partial P_z^- / \partial t + \Phi_z^- A_z^-; \\
\alpha_y^+ \langle u \rangle \left| \partial P_y^+ / \partial y \right| &= m F_y^+ \partial P_y^+ / \partial t + \Phi_y^+ A_y^+; \\
\alpha_y^- \langle u \rangle \left| \partial P_y^- / \partial y \right| &= m F_y^- \partial P_y^- / \partial t + \Phi_y^- A_y^-,
\end{aligned}
\tag{10}
$$

$$
\alpha_z^{+,-} = \int_0^\infty \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 r^2 u \sin\varphi \, dr \, d\varphi \, d\psi \times \left[\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 r^2 u \cos\varphi \cos\psi \, dr \, d\varphi \, d\psi \right]^{-1};
$$

$$
A = A_z^+ + A_z^- = A_y^+ + A_y^-, \qquad A_z^{+,-} = 2mB^{-1} \int_0^\infty \int_0^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 r q \, dr \, d\varphi \, d\psi,
$$

$$
\alpha_y^{+,-} = \int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} f f_1 f_2 r^2 u \cos\varphi \sin\psi \, dr \, d\varphi \, d\psi \times \left[\int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} f f_1 f_2 r^2 u \cos\varphi \cos\psi \, dr \, d\varphi \, d\psi \right]^{-1};
$$

$$
A_y^{+,-} = 2mB^{-1} \int_0^\infty \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} f f_1 f_2 r q \, dr \, d\varphi \, d\psi,
$$

where the quantities $\Phi_z^{+,-}$, $\Phi_y^{+,-}$ are introduced analogously to how the quantity Φ_x was introduced in equation (4). Subtracting pairwise from one another the equations for the z-direction and for the y-direction, using relations (5) and (6), and adopting for simplicity the conditions $\alpha_z^+ = \alpha_z^- = \frac{1}{2}\alpha_z$, $\alpha_y^+ = \alpha_y^- = \frac{1}{2}\alpha_y$, $F_z^+ = F_z^- = \frac{1}{2}F_z$, $F_y^+ = F_y^- = \frac{1}{2}F_y$, $\Phi_z^+ = \Phi_z^- = \frac{1}{2}\Phi_z$, $\Phi_y^+ = \Phi_y^- = \frac{1}{2}\Phi_y$, we obtain

$$
\alpha_z \langle u \rangle \left| \frac{\partial P}{\partial z} \right| = m F_z \frac{\partial}{\partial t} (P_z^+ - P_z^-) + \Phi_z (A_z^+ - A_z^-); \tag{11}
$$

$$
\alpha_y \langle u \rangle \left| \frac{\partial P}{\partial y} \right| = m F_y \frac{\partial}{\partial t} (P_y^+ - P_y^-) + \Phi_y (A_y^+ - A_y^-). \tag{12}
$$

To relate the quantities A^+, A^- to P^+, P^-, we use the sorption equation ([2,3]), according to which $A = \beta(P - P_*)$, and, consequently, $A_z^+ = \beta(P_z^+ - \frac{1}{2}P_*)$, $A_z^- = \beta(P_z^- - \frac{1}{2}P_*)$, $A_y^+ = \beta(P_y^+ - \frac{1}{2}P_*)$, $A_y^- = \beta(P_y^- - \frac{1}{2}P_*)$, where β is a certain kinetic coefficient, and P_* is the concentration of the impurity, averaged for a fixed point, that is in equilibrium with the impurity sorbed on the walls of the pore channels. Expressing the differences $(P_z^+ - P_z^-)$, $(P_y^+ - P_y^-)$ in equations

(11), (12) through the quantities Q_z and Q_y in accordance with their definition by the formulas $Q_z = \alpha_z\langle u\rangle(P_z^+ - P_z^-)$, $Q_y = \alpha_y\langle u\rangle(P_y^+ - P_y^-)$, we finally obtain

$$Q_z = -\frac{\alpha_z^2}{\beta\Phi_z}\langle u\rangle^2\frac{\partial P}{\partial z} - \frac{F_z}{\beta\Phi_z}\frac{\partial Q_z}{\partial t},$$

$$Q_y = -\frac{\alpha_y^2}{\beta\Phi_y}\langle u\rangle^2\frac{\partial P}{\partial y} - \frac{F_y}{\beta\Phi_y}\frac{\partial Q_y}{\partial t}.$$

This result is in agreement with dimensional considerations. Indeed, the condition $A = \text{const}$ means, in accordance with the sorption isotherm, that $\partial P_*/\partial t \neq 0$, and, consequently, $\partial P/\partial t \neq 0$. Therefore, in order to determine from dimensional considerations the magnitude of the effective mixing coefficient, which has the dimension of the square of length divided by time, one should take, as the defining dimensional characteristic, along with the quantity $\langle u\rangle$, a parameter having the dimension of time. Such a parameter is the kinetic coefficient β^{-1}. The combination of $\langle u\rangle$ and β^{-1} gives, for the effective mixing coefficient, a quantity proportional to $\langle u\rangle^2\beta^{-1}$.

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Note: Figure translations are in progress. See original paper for figures.