ON THE THEORY OF SOLVABLE LINEAR GROUPS

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Abstract

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§ 1. Let Δ be an arbitrary field, and let G be a maximal irreducible primitive solvable subgroup of $GL(n, \Delta)$. The structure of the group G is conveniently studied by means of the invariant series

$$G \supseteq V \supseteq A \supseteq F \supset (e), \tag{1}$$

where F is a maximal Abelian normal divisor of the group G; V is the centralizer of F in G; A/F is a maximal subgroup among the Abelian invariant subgroups of G/F contained in V/F.

Below we shall need the following known properties of the series (1).

1) G has a unique maximal Abelian normal divisor F (^{4,5}).
2) F is the multiplicative group Σ^* of some extension Σ of the field Δ, and the degree $m = \Sigma : \Delta$ divides the number n, while char Δ does not divide the number $r = n/m$ (^{4}).
3) The order of the group A/F is equal to r^2; the Sylow subgroups of A/F are elementary Abelian groups; the linear Δ-envelope of the group A coincides with the algebra Σ_r (^{3,4}).
4) If aF is an element of order δ of the factor group A/F, then the index of the centralizer of the element a in A is equal to δ (^{4}). If B is such a subgroup of A that $B \supset F$, then the rank of the linear Σ-envelope of the group B over Σ coincides with the order of B/F (^{4}).
5) Let p be a prime divisor of the number r, and let B/F be a p-subgroup of A/F, invariant in G/F. Then

$$B = (c_1)(d_1) \dots (c_\nu)(d_\nu)F,$$

$$(c_j, d_j) = c_{jd_{jc}}j^{-1}d_j^{-1} = \eta; \quad j = 1, \dots, \nu;$$

$$(c_i, c_j) = (d_i, d_j) = 1; \quad i \neq j \Rightarrow (c_i, d_j) = 1, \tag{2}$$

where η is an element of order p in F. The order of B/F is equal to $p^{2\nu}$ ([4]).

6) Let $r = p_1^{l_1} \cdots p_k^{l_k}$ be the canonical decomposition of the number r, and let P_i/F be the Sylow p_i-subgroup of A/F. Then for $i \neq j$ the mutual commutator (P_i, P_j) is equal to 1. Obviously, the order of P_i/F is equal to $p_i^{2l_i}$, and P_i is a normal divisor of G. On solvable linear groups see also ([1,2,6]). On the properties of A see ([7]).

In the present paper it is proved that, for an arbitrary field Δ, the group A of the series (1) is uniquely determined by the group G. For an algebraically closed field Ω it is shown that the description of maximal irreducible primitive solvable subgroups of $GL(n, \Omega)$ reduces to the case when n is a power of a prime number. A one-to-one correspondence is established between the conjugacy classes of maximal irreducible primitive solvable subgroups of $GL(p^l, \Omega)$ (p a prime number $\neq \operatorname{char} \Omega$) and the conjugacy classes of maximal s-irreducible (see the definition below) solvable subgroups of the symplectic group $Sp(2l, p)$.

§ 2. **Lemma 1.** Let P/F be a p-subgroup of a Sylow group of A/F, and let B/F be a subgroup of the group P/F. Then the index of the centralizer of B in P is equal to the order of the group B/F.

Lemma 2. Let B/F be a normal divisor of G/F contained in P/F, where P/F is a Sylow p-subgroup of A/F. Then the group P/F is representable

in the form of a direct product $P/F = B/F \cdot C/F$, where C/F is a normal divisor of G/F such that $(B, C) = 1$.

Proof. Let C be the centralizer of B in P. Since B and P are normal divisors of G, C is also a normal divisor of G. The center of the group B coincides with F; hence $B \cap C = F$, i.e. $B/F \cap C/F = F/F$. By Lemma 1, $P : C = B : F$. Since $P : F = (P : C)(C : F)$, we have $(B : F)(C : F) = P : F$. It follows from this and from the preceding that P/F is the direct product of its subgroups B/F and C/F.

§ 3. **Theorem 1.** In the group G there is only one subgroup A such that: (I) A/F is an abelian normal divisor of G/F; (II) $A/F \subseteq V/F$; (III) A/F is maximal among the subgroups of G/F having properties (I) and (II).

Proof. Let A and B be such subgroups of G that A/F and B/F have properties (I), (II), (III). We shall show that $A = B$. Let p be an arbitrary prime divisor of the number r, let P/F be a Sylow p-subgroup of A/F, and let Q/F be a Sylow p-subgroup of B/F. Put $D/F = P/F \cap Q/F$. Since D/F is a normal divisor of G/F, by Lemma 2 $P/F = D/F \cdot U/F$, $Q/F = D/F \cdot W/F$, where the direct factors U/F and W/F are normal divisors of G/F, and $(D, U) = (D, W) = 1$. Consequently, UW/F is an abelian normal divisor of G/F. $D/F \cdot UW/F = C/F$ is also an abelian normal divisor of G/F, and $C/F \subseteq V/F$. Obviously,

$C/F \supseteq P/F,\ C/F \supseteq Q/F$. In view of (III), $C/F = P/F,\ C/F = Q/F$. Hence $P = Q,\ A = B$. The theorem is proved.

Theorem 2. Let A be a group from the series (1); let $\mathfrak{N}$ be the normalizer of A in $GL(n, \Delta)$; and let G_1 be a subgroup of $GL(n, \Delta)$ such that $A \lhd G_1$. If G_1 is conjugate to G in $GL(n, \Delta)$, then it is conjugate to G in $\mathfrak{N}$.

Proof. Let $G_1 = dGd^{-1}$, $d \in GL(n, \Delta)$. Since F is the center of A and $A \lhd G_1$, F is an abelian normal divisor of G_1, and $d^{-1}Fd$ is an abelian normal divisor of G. By property 1) of the series (1), $d^{-1}Fd \subseteq F$. Since $F = \Sigma^*$, where Σ is an extension of degree m of the field Δ, we have $d^{-1}\Sigma d = \Sigma$, $d^{-1}Fd = F$. Consequently, F is a maximal abelian normal divisor of G_1. Since $A : F = r^2$, A/F is maximal among the abelian invariant subgroups of G_1/F contained in V_1/F, where V_1 is the centralizer of F in G_1. The group dAd^{-1}/F has the same properties. By Theorem 1, $dAd^{-1} = A$, i.e. $d \in \mathfrak{N}$. The theorem is proved.

§ 4. We now turn to the subgroups P_i of the group A (see 6)). The group A and its subgroups P_i will be regarded as subgroups of $GL(r, \Sigma)$ (see property 3)).

Lemma 3. The irreducible components of the group P_j are pairwise equivalent and absolutely irreducible; their degrees are equal to the number $p_j^{l_j}$ (see 6)).

Lemma 4. In a suitable basis of the space Σ^r, the matrices a of the group A take the form

$$a = u_1 \times \cdots \times u_k, \tag{3}$$

$GL(p_j^{l_j}, \Sigma)$, isomorphic to the group P_j, and $\times$ denotes the Kronecker product of matrices. P_j consists of matrices of the form

$$E_{p_1^{l_1}} \times \cdots \times u_j \times \cdots \times E_{p_k^{l_k}}, \quad u_j \in P^j. \tag{4}$$

Lemma 5. Let H be the normalizer of A in $GL(r, \Sigma)$, and H_j the normalizer of P_j in $GL(p_j^{l_j}, \Sigma)$. Then H consists of all matrices of the form

$$h = h_1 \times \cdots \times h_k, \tag{5}$$

where $h_j \in H_j$.

Proof. Since P_j is a normal divisor of the group H, for $u_j \in P^j,\ h \in H$, by virtue of (4) we have

$$h \left(E_{p_1^{l_1}} \times \cdots \times u_j \times \cdots \times E_{p_k^{l_k}} \right) h^{-1} = E_{p_1^{l_1}} \times \cdots \times u_j' \times \cdots \times E_{p_k^{l_k}},$$

where $u_j' \in P^j$. Obviously, the mapping

$$u_j \to u_j' \tag{6}$$

is an automorphism of the group P^j. If $c, d \in P_j$, $\lambda \in \Sigma$, then $h(c+d)h^{-1} = hch^{-1} + hdh^{-1}$, $h\lambda ch^{-1} = \lambda hch^{-1}$. Consequently, the mapping (6) can be extended to an automorphism φ_j of the linear Σ-envelope $[P^j]$ of the group P^j. By Lemma 3, $[P^j]$ coincides with the algebra $\Sigma_{p^j}^{l_j}$. Hence

$$\varphi_j(u_j) = h_j u_j h_j^{-1}, \quad h_j \in GL(p^{l_j}, \Sigma), \quad h_j \in H_j.$$

Thus, for any $h \in H$ there are $h_1, \dots, h_k$, where $h_j \in H_j$, such that

$$hxh^{-1} = gxg^{-1}, \quad g = h_1 \times \cdots \times h_k$$

for every $x \in A$. It follows that

$$h^{-1}g = f \in F, \quad h = \sigma h_1 \times \cdots \times h_k, \quad \sigma \in \Sigma^*.$$

Obviously, $\sigma h_1 \in H_1$.

Theorem 3. Let Ω be an algebraically closed field, and let

$$n = p_1^{l_1} \cdots p_k^{l_k}$$

be the canonical factorization of the number n, with $\operatorname{char}\Omega$ not dividing the number n. Then every maximal irreducible primitive solvable subgroup G of the group $GL(n, \Omega)$, in a suitable basis of Ω^n, is representable in the form

$$G = G_1 \times \cdots \times G_k,$$

where G_j is a maximal irreducible primitive solvable subgroup of

$$GL(p_j^{l_j}, \Omega), \quad j = 1, \dots, k,$$

and × denotes the Kronecker-product sign.

§ 5. Let now $\Delta = \Omega$, and let $n = p^l$, where Ω is an algebraically closed field and p is a prime number ($p \neq \operatorname{char}\Omega$). Then, in a suitable basis of Ω^n, the group A of the series (1) has the form (4)

$$A = (a_1)(b_1) \cdots (a_l)(b_l)F, \tag{7}$$

$$a_j = E_{p^{j-1}} \times c \times E_{p^{l-j}}, \quad b_j = E_{p^{j-1}} \times d \times E_{p^{l-j}}, \quad j = 1, \dots, l,$$

where

$$c = \operatorname{diag}[1, \eta, \dots, \eta^{p-1}], \quad \eta \in \Omega, \quad \eta^p = 1, \quad \eta \neq 1,$$

$$d = \begin{bmatrix} 00 \dots 01 \\ 10 \dots 00 \\ 01 \dots 00 \\ \dots \\ 00 \dots 10 \end{bmatrix}.$$

Consider the normalizer $\mathfrak{N}$ of the group A in $GL(p^l, \Omega)$. For $x \in \mathfrak{N}$,

$$xa_{jx}^{-1} = a'_j = \lambda_j a_1^{\alpha_{1j}} \cdots a_l^{\alpha_{lj}} b_1^{\gamma_{1j}} \cdots b_l^{\gamma_{lj}},$$

$$xb_{jx}^{-1} = b'_j = \mu_j a_1^{\beta_{1j}} \cdots a_l^{\beta_{lj}} b_1^{\delta_{1j}} \cdots b_l^{\delta_{lj}}, \tag{8}$$

$$\lambda_j, \mu_j \in \Omega^*; \quad \alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in GF(p),$$

and a_j and b_j satisfy the conditions

$$(a_i, b_i) = \eta; \quad (a_i, a_\mu) = 1; \quad (b_i, b_\mu) = 1; \quad i \neq \mu \Rightarrow (a_i, b_\mu) = 1. \tag{9}$$

By virtue of (8), a'_j and b'_j satisfy the same conditions:

$$(a'_i, b'_i) = \eta; \quad (a'_i, a'_\mu) = 1; \quad (b'_i, b'_\mu) = 1; \quad i \neq \mu \Rightarrow (a'_i, b'_\mu) = 1. \tag{10}$$

Put

$$\psi(x) = \begin{bmatrix} \alpha_{11} \ldots \alpha_{1l} \beta_{11} \ldots \beta_{1l} \\ \cdots \\ \alpha_{l1} \ldots \alpha_{ll} \beta_{l1} \ldots \beta_{ll} \\ \gamma_{11} \ldots \gamma_{1l} \delta_{11} \ldots \delta_{1l} \\ \cdots \\ \gamma_{l1} \ldots \gamma_{ll} \delta_{l1} \ldots \delta_{ll} \end{bmatrix}, \quad x \in \mathfrak{N}. \tag{11}$$

Conditions (10) are equivalent to the following equalities:

$$1. \quad \sum_{\rho=1}^{l} (\alpha_{\rho i} \delta_{\rho i} - \beta_{\rho i} \gamma_{\rho i}) = 1.$$

$$2. \quad \sum_{\rho=1}^{l} (\alpha_{\rho i} \gamma_{\rho \mu} - \alpha_{\rho \mu} \gamma_{\rho i}) = 0. \tag{12}$$

3.

$$\sum_{\rho=1}^{l} (\beta_{\rho i} \delta_{\rho \mu} - \beta_{\rho \mu} \delta_{\rho i}) = 0. \tag{12}$$

4. $\mu \neq i \Rightarrow$

$$\sum_{\rho=1}^{l} (\alpha_{\rho i} \delta_{\rho \mu} - \beta_{\rho \mu} \gamma_{\rho i}) = 0.$$

We introduce the symplectic group $\mathrm{Sp}(2l, p)$ over $GF(p)$ in the following way. Let

$$\Phi_l = \begin{bmatrix} 0 & E_l \\ -E_l & 0 \end{bmatrix}.$$

The symplectic group $\mathrm{Sp}(2l, p)$ will be the group of all matrices h in $GL(2l, p)$ satisfying the condition

$$ {}^t h \Phi_l h = \Phi_l, \tag{13}$$

where ${}^t h$ is the transpose of h.

Lemma 6. The elements of the matrix

$$h = \begin{bmatrix} \alpha_{11} \dots \alpha_{1l} \beta_{11} \dots \beta_{1l} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \alpha_{l1} \dots \alpha_{ll} \beta_{l1} \dots \beta_{ll} \\ \gamma_{11} \dots \gamma_{1l} \delta_{11} \dots \delta_{1l} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \gamma_{l1} \dots \gamma_{ll} \delta_{l1} \dots \delta_{ll} \end{bmatrix}$$

over $GF(p)$ satisfy conditions (12) if and only if $h \in \mathrm{Sp}(2l, p)$.

Lemma 7. $\mathrm{Sp}(2l, p) \simeq \mathfrak{R}/A$, for the mapping

$$\psi : \mathfrak{R} \to \mathrm{Sp}(2l, p),$$

defined by formula (11), is an epimorphism whose kernel is the group A.

Lemma 8. Let G_1 and G_2 be such maximal irreducible primitive solvable subgroups of $GL(p^l, \Omega)$ that $A \lhd G_1$, $A \lhd G_2$. Then G_1 and G_2 are conjugate in $GL(p^l, \Omega)$ if and only if $H_1 = \psi(G_1)$ and $H_2 = \psi(G_2)$ are conjugate in $\mathrm{Sp}(2l, p)$.

A subgroup H of the group $\mathrm{Sp}(2l, p)$ will be called s-reducible if there exist an integer ν, $1 \leq \nu \leq l$, and a matrix $t \in \mathrm{Sp}(2l, p)$ such that the matrices of the group $t^{-1} H t$ have the form

$$\begin{bmatrix} \alpha_{11} \dots \alpha_{1\nu} & * \\ \cdot\;\cdot\;\cdot\;\cdot & \\ \alpha_{\nu 1} \dots \alpha_{\nu\nu} & \\ 0 \dots 0 & \\ \cdot\;\cdot\;\cdot\;\cdot & * \\ 0 \dots 0 & \end{bmatrix}.$$

In the opposite case the subgroup H of the group $\mathrm{Sp}(2l, p)$ is called s-irreducible. The following is true.

Lemma 9. Let H be a subgroup of $\mathrm{Sp}(2l, p)$. The subgroup $\psi^{-1}(H)$ of the group $GL(p^l, \Omega)$ is primitive if and only if H is irreducible.

From the last two lemmas it follows:

Theorem 4. Let $H_1, \dots, H_\rho$ be maximal solvable s-irreducible subgroups of $\mathrm{Sp}(2l, p)$, and let every maximal solvable s-irreducible subgroup of $\mathrm{Sp}(2l, p)$ be conjugate in $\mathrm{Sp}(2l, p)$ with one and only one H_j. Then

$$\psi^{-1}(H_1), \dots, \psi^{-1}(H_\rho)$$

are maximal solvable irreducible primitive subgroups of $GL(p^l, \Omega)$, and every maximal solvable irreducible primitive subgroup of $GL(p^l, \Omega)$ is conjugate with one and only one $\psi^{-1}(H_j)$.

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