VARIABLE PLASTICITY AND ACCUMULATION OF DAMAGEABILITY

1969

SovietRxiv

Abstract

Full Text

UDC 539.374.1

THEORY OF ELASTICITY

V. V. MOSKVITIN

VARIABLE PLASTICITY AND ACCUMULATION OF DAMAGEABILITY

(Presented by Academician Yu. N. Rabotnov on 28 V 1968)

Existing methods for estimating variable loading in the presence of finite regions of plastic deformation have a certain limitation. This limitation consists in the fact that the corresponding equations of state for a given medium contain, as a rule, only one parameter—the number of loadings. Meanwhile, the elastic-plastic properties of the material will also depend on the amplitudes of the stresses and strains of the preceding variable loadings. On the other hand, the degree of accumulation of damageability, developing in media under their variable loading and leading to loss of strength, must also depend on the stress amplitude. This circumstance naturally suggests the use of a damageability function for estimating the elastic-plastic properties of a material subjected to variable loadings and for establishing a condition of cyclic strength.

Below a model is constructed of an elastic-plastic medium whose physicomechanical properties depend on the degree of accumulated damageability. Equations of the type of the deformation theory and of the flow theory are used. A system of experiments is presented for determining the material functions.

§ 1. **Statement of the problem.** Denote by $\sigma_{ij}^{(n)}$ and $\varepsilon_{ij}^{(n)}$ the components of the stress and strain tensors under the n-th loading, and let the beginning of the n-th half-cycle coincide with the beginning of the $(n-1)$-st unloading. Introduce the differences

$$\overline{\sigma}_{ij}^{(n)} = (-1)^n \left(\sigma_{ij}^{(n-1)} - \sigma_{ij}^{(n)}\right), \qquad \overline{\varepsilon}_{ij}^{(n)} = (-1)^n \left(\varepsilon_{ij}^{(n-1)} - \varepsilon_{ij}^{(n)}\right) \tag{1.1}$$

$$(i, j = 1, 2, 3,\ n \geqslant 1,\ \sigma_{ij}^{(0)} = 0,\ \varepsilon_{ij}^{(0)} = 0).$$

The quantities $\overline{\sigma}_{ij}^{(n)}$ and $\overline{\varepsilon}_{ij}^{(n)}$ have the following property: at the beginning of the n-th loading they are equal to zero.

Let us first use the laws of the theory of small elastic-plastic strains (^{1}) to establish the relation between stresses and strains under variable loadings (^{2})

$$\overline{S}_{ij}^{(n)} = \frac{2\overline{\sigma}_u^{(n)}}{3\overline{\varepsilon}_u^{(n)}}{}_{ij}^{(n)}, \qquad \overline{\sigma}_u^{(n)} = \left({}^3/{}_2\overline{S}_{ij}^{(n)}\overline{S}_{ij}^{(n)}\right)^{1/2}, \qquad \overline{\varepsilon}_u^{(n)} = \left({}^2/{}_3{}_{ij}^{(n)}{}_{ij}^{(n)}\right)^{1/2}, \qquad \overline{\sigma}^{(n)} = K\overline{\theta}^{(n)}. \tag{1.2}$$

In order that the system of equations relating stresses and strains be closed, one must also introduce a function relating the invariant quantities $\overline{\sigma}_u^{(n)}$ and $\overline{\varepsilon}_u^{(n)}$. In contrast to the approaches that have been developed up to now, we shall assume that $\overline{\sigma}_u^{(n)}$ is a universal function of $\overline{\varepsilon}_u^{(n)}$ and also of some function $0 \leqslant \omega \leqslant 1$, characterizing the degree of damageability of the material,

$$\overline{\sigma}_u^{(n)} = \Phi_n\left[\overline{\varepsilon}_u^{(n)}, \omega(n-1)\right], \qquad \overline{\varepsilon}_u^{(n)} \gg \overline{\varepsilon}_s^{(n)}, \qquad \overline{\sigma}_u^{(n)} = 3G\overline{\varepsilon}_u^{(n)}, \qquad \overline{\varepsilon}_u^{(n)} \ll \overline{\varepsilon}_s^{(n)}. \tag{1.3}$$

The concept of a damage function was introduced in the literature in connection with the study of creep and long-term strength by L. M. Kachanov (^{3,4}), Yu. N. Rabotnov ($^{5-7}$), A. A. Ilyushin (^{8}), and others. Without dwelling here on the physical essence of the function ω, we introduce the kinetic equations

$$\partial\omega/\partial n = f(\omega, \overline{\sigma}_u^{(n)}), \tag{1,4}$$

$$\omega = \int_0^n F(n-m)\,\varphi(\overline{\sigma}_u^{(m)})\,dm, \tag{1,5}$$

where either equation (1,4) or equation (1,5) holds. By the quantities $\overline{\sigma}_u^{(n)}$ entering equations (1,4) and (1,5), we shall understand their maximum values under the n-th loading. Failure corresponds to the value $\omega(n_b) = 1$, where n_b is the number of loadings until the limiting state is reached.

The functions Φ and f, or Φ, F, and φ, are determined experimentally. Their choice makes it possible to describe various experimental observations. Examples of determining these functions are given in § 2.

Equations (1,2), (1,3), and (1,4), or (1,5), completely determine the relation between stresses and strains; they take into account the influence of the degree of damage on the elastic-plastic properties of the material and contain within themselves a strength criterion under variable loadings. If $\overline{\sigma}_{ij}^{(k)}$, $\overline{\varepsilon}_{ij}^{(k)}$ are known for any $k = 2, 3, 4, \ldots$, the desired stresses $\sigma_{ij}^{(n)}$ and strains $\varepsilon_{ij}^{(n)}$ are determined by the formulas

$$\sigma_{ij}^{(n)} = \sigma_{ij}^{(1)} + \sum_{k=2}^{n}(-1)^{k-1}\overline{\sigma}_{ij}^{(k)}, \qquad \varepsilon_{ij}^{(n)} = \varepsilon_{ij}^{(1)} + \sum_{k=2}^{n}(-1)^{k-1}\overline{\varepsilon}_{ij}^{(k)}. \tag{1,6}$$

The equations given above are constructed within the framework of the deformation theory. We shall now use the simplest flow theory ($^{9-11}$) and write the relation between the stresses $\sigma_{ij}^{(n)}$ and strains $\varepsilon_{ij}^{(n)}$ for the n-th loading:

$$d\mathcal{E}_{ij}^{(n)} = \frac{dS_{ij}^{(n)}}{2G} + \frac{\partial f_n}{\partial S_{ij}^{(n)}}\, d\lambda_n, \qquad \sigma^{(n)} = 3K\varepsilon^{(n)},$$

$$f_n = (S_{ij}^{(n)} - H_n e_{ij}^{(n)})(S_{ij}^{(n)} - H_n e_{ij}^{(n)}) - k_n^2 = 0. \tag{1,7}$$

In contrast to existing approaches, we shall assume that the quantities H_n and k_n are experimental functions of the degree of accumulated damage ω. Methods for determining these functions are given in § 2.

In analogous fashion, other variants of flow theory (^{12,13}) and the equations of the general mathematical theory of plasticity (^{14}) may also be used, with the introduction of an additional argument for the parameters and functions of the invariants. To the equations relating stresses and strains there is added a kinetic equation of the form (1,4) or (1,5).

Finally, equations of type (1,2) or (1,7) must be supplemented by the differential equations of equilibrium, the equations of strain compatibility, and boundary conditions.

In conclusion, let us note the following. In the formulation presented above it was assumed that the elastic constants do not change their values in the process of variable plastic deformation. However, this change can be taken into account if the matrix of elastic constants is regarded as a function of the degree of accumulated damage ω.

§ 2. **Determination of material functions.** We present a system of experiments for determining the functions introduced in § 1.

Let us first consider equation (1,4) and represent the function f in the form

$$f = A(\overline{\sigma}_u^{(n)})^\alpha/(1-\gamma\omega)^\beta, \tag{2,1}$$

where A, α, β, γ are material constants. From (1.4) it follows that

$$1-(1-\gamma\omega)^{\beta+1} = A\gamma(\beta+1)\int_0^n \left(\bar{\sigma}_u^{(n)}\right)^\alpha dn. \tag{2.2}$$

Here it has been taken into account that at $n=0$, $\omega=0$.

Let an experiment be carried out with cyclic loading of specimens (tension–compression) with a stress amplitude $\bar{\sigma}_1$ independent of the number of loadings n. Since in this case $\bar{\sigma}_u^{(n)} = \bar{\sigma}_1 = \text{const}$, from (2.2) it follows that

$$(1-\gamma\omega)^{\beta+1} = 1 - A\gamma(\beta+1)\bar{\sigma}_1^{-\alpha} n. \tag{2.3}$$

Since at $n = n_b$, $\omega = 1$, from relation (2.3) we obtain

$$n_b \bar{\sigma}_1^{\alpha} = B, \qquad B = \left[1-(1-\gamma)^{\beta+1}\right]/A\gamma(\beta+1), \tag{2.4}$$

which coincides exactly with the experimental data available in the literature (see, for example, [15]). Thus, from the experiment two constants α and B become known. If one takes $\gamma = 1$ and $\beta = \alpha$, then function (2.1) becomes known, since in this case $AB(\alpha+1) = 1$. In this case from (2.3) it follows that

$$(1-\omega)^{\alpha+1} = 1 - \bar{\sigma}_1^{\alpha} n/B. \tag{2.5}$$

It is clear that, in the general case as well, no experiments on cyclic strength can make it possible to determine any constants other than α and B.

Let us now consider equation (1.5), and let us assume that the constants α and B in relations (2.4) are known from experiment. Since, as before, $\bar{\sigma}_u^{(n)} = \bar{\sigma}_1 =$ const, from (1.5) and (2.4), under the condition $\varphi(\bar{\sigma}_1) = \bar{\sigma}_1^d$, it follows that

$$F(n-m) = \frac{d}{\alpha B^{d/\alpha}}(n-m)^{-1+d/\alpha}, \qquad \omega = \frac{d}{\alpha B^{d/\alpha}}\int_0^n (n-m)^{-1+d/\alpha}\left(\bar{\sigma}_u^{(m)}\right)^d dm. \tag{2.6}$$

The unknown constant d can be determined from an additional experiment, for example, an experiment with cyclic loading with an amplitude increasing proportionally to the number of loadings, $\bar{\sigma}_1^{(m)} = k^2 m$. In this case the equation for determining the value d will be

$$\alpha B^{d/\alpha} = dk^2 \int_0^{n_b} (n_b - m)^{-1+d/\alpha} m^d dm.$$

Let us note that from relation (2.6), for $\bar{\sigma}_u^{(m)} = \bar{\sigma}_1$, we have

$$\omega B^{d/\alpha} = \bar{\sigma}_1^d n^{d/\alpha}. \tag{2.7}$$

Finally, let us consider the question of determining function (1.3). Suppose that, as a result of experiments (cyclic tension—compression) at $\sigma_1^{(n)} = \bar{\sigma}_1 =$ const, the amplitudes of deformation $\bar{\varepsilon}_1^{(n)}(\bar{\sigma}_1)$, the stresses $\bar{\sigma}_s^{(n)}(\bar{\sigma}_1)$, and the strains $\bar{\varepsilon}_s^{(n)}(\bar{\sigma}_1)$ corresponding to the yield limit have been found, and, in addition, the strains $\bar{\varepsilon}_{a_i}^{(n)}(\bar{\sigma}_1)$ corresponding to the stresses $\alpha_i \bar{\sigma}_1 > \bar{\sigma}_s^{(n)}$ $(0 < \alpha_i < 1,\ i = 1, 2, ...)$ have been found.

For definiteness, let us represent function (1.3) in the form of a polynomial of p-th order

$$\frac{\bar{\sigma}_u^{(n)}}{\bar{\sigma}_s^{(n)}} = \sum_{k=0}^{p} a_k(\omega, n) \left(\frac{\bar{\varepsilon}_u^{(n)}}{\bar{\varepsilon}_s^{(n)}} \right)^k, \qquad \sum_{k=0}^{p} a_k = 1. \tag{2.8}$$

Hence, for the experiment indicated above, in the case of incompressibility,

$$\frac{\alpha_i \bar{\sigma}_1}{\bar{\sigma}_s^{(n)}(\bar{\sigma}_1)} = \sum_{k=0}^{p} a_k \left(\frac{\bar{\varepsilon}_{\alpha_i}^{(n)}(\bar{\sigma}_1)}{\bar{\varepsilon}_s^{(n)}(\bar{\sigma}_1)} \right)^k \qquad (0 < \alpha_i \leqslant 1). \tag{2.9}$$

Equations (2.9) contain the only unknown coefficients a_k, which may be determined, for example, by approximating functions by the method of least squares. The quantities a_k will now be known functions of the stress $\bar{\sigma}_1$ and of the numbers n, i.e. $a_k = a_k(\bar{\sigma}_1, n)$. Having determined, in turn, $\bar{\sigma}_1$ from equation (2.5) or from equation (2.7), replacing n by $n-1$, and substituting the value of $\bar{\sigma}_1$ thus found into the expression a_k, we find the desired coefficients $a_k = a_k(\omega, n)$.

In the case of linear hardening,

$$\bar{\sigma}_u^{(n)} = \bar{\sigma}_s^{(n)}(\omega) a(\omega, n) + 3G[1 - a(\omega, n)] \bar{\varepsilon}_u^{(n)},$$

where, until the appearance of plastic strains, $a(\omega, n) = 0$.

From this it is easy to pass to the determination of the parameter $\bar{H}_n$ appearing in the relations of the flow theory (1.7):

$$\bar{H}_n(\omega) = 2G[1 - a(\omega, n)]/a(\omega, n). \tag{2.10}$$

As for the parameter k_n, it is easy to see that it is equal to $\sqrt{6\bar{k}_n} = \bar{\sigma}_s^{(n)}(\omega)$.

For the case of the particular model of a linearly hardening medium recently proposed in paper [16],

$$a = \lambda = \text{const}, \qquad \bar{\sigma}_s^{(n)} = \alpha_1^{n-2}(1 + \alpha_1)\sigma_s + \bar{\sigma}_1 \left(1 - \frac{1 + \alpha_1}{2} \alpha_1^{n-2} \right).$$

In this case, taking (2.5) into account,

$$\bar{\sigma}_u^{(n)} = \lambda \alpha_1^{n-2}(1 + \alpha_1)\sigma_s + \lambda \left(1 - \frac{1 + \alpha_1}{2} \alpha_1^{n-2} \right) [B - B(1 - \omega)^{1+\alpha}]^{1/\alpha} \times$$

$$\times(n-1)^{-1/\alpha}+3G(1-\lambda)\bar{\varepsilon}_u^{(n)},$$

$$\bar{H}_n=2G\frac{1-\lambda}{\lambda}=\text{const},$$

$$\sqrt{6\bar{k}_n}=\alpha_1^{(n-2)}(1+\alpha_1)\sigma_s+\left(1-\frac{1+\alpha_1}{2}\alpha_1^{n-2}\right)\left[B-B(1-\omega)^{1+\alpha}\right]^{1/\alpha}(n-1)^{-1/\alpha}.$$

Moscow State University
named after M. V. Lomonosov

Received
24 V 1968

CITED LITERATURE

[1] A. A. Il' yushin, *Plasticity*, Part I, Moscow-Leningrad, 1948.
[2] V. V. Moskvitin, *Plasticity under Variable Loadings*, Moscow State University Press, 1965.
[3] L. M. Kachanov, Izv. AN SSSR, OTN, No. 8 (1958).
[4] L. M. Kachanov, *Theory of Creep*, Moscow, 1960.
[5] Yu. N. Rabotnov, in: *Problems of the Strength of Materials and Structures*, Academy of Sciences of the USSR Press, 1959.
[6] Yu. N. Rabotnov, Prikl. mekh. i tekhn. fiz., No. 2 (1963).
[7] Yu. N. Rabotnov, *Creep of Structural Elements*, "Nauka," 1966.
[8] A. A. Il' yushin, Inzh. zhurn., Mekhanika tverd. tela, No. 3, 21 (1967).
[9] V. D. Klyushnikov, PMM, 22, issue 1 (1958).
[10] A. Yu. Ishlinskii, Ukr. matem. zhurn., 6, No. 3 (1954).
[11] W. Prager, J. Appl. Mech., 23, No. 4, 493 (1956).
[12] Yu. I. Kadash vich, V. V. Novozhilov, DAN, 117, No. 4 (1957).
[13] R. A. Arutyunyan, A. A. Vakulenko, Izv. AN SSSR, Mekhanika, No. 4 (1965).
[14] A. A. Il' yushin, *Plasticity*, Academy of Sciences of the USSR Press, 1963.
[15] G. R. Halford, J. Morrow, Proc. Am. Soc. for Testing Mater., 62 (1962).
[16] N. I. Chernyak, D. A. Gavrilov, Prikl. mekh., Academy of Sciences of the Ukrainian SSR Press, No. 9 (1968).