ON A REPRESENTATION OF THE LAPLACE TRANSFORM OF THE FUNCTION \(\exp(-t^2)\operatorname{erf}(at)\)

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Abstract

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ON A REPRESENTATION OF THE LAPLACE TRANSFORM OF THE FUNCTION $\exp(-t^2)\,\mathrm{erf}(at)$

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In the present paper, for the integral

$$\int_0^\infty \exp(-pt - t^2)\,\mathrm{erf}(at)\,dt, \tag{1}$$

where p and a are certain complex numbers, a representation is given that makes it possible to obtain many formulas very simply and to express (1) in closed form in terms of known special functions.

Theorem. Let $\mathrm{Re}\,a^2 > -1$, p be an arbitrary complex number if $\mathrm{Re}\,a \neq 0$, and $\mathrm{Re}\,p > 0$ if a is purely imaginary. Then the following equality holds:

$$\int_0^\infty \exp(-pt - t^2)\,\mathrm{erf}(at)\,dt = \frac{a\exp(p^2/4)}{2\sqrt{1+a^2}}\int_0^\infty \exp\left(-\frac{a^2t^2}{4(1+a^2)}\right)\mathrm{erfc}\left(\frac{t}{2\sqrt{1+a^2}}\right)dt. \tag{2}$$

Proof. In view of the asymptotic properties of the probability integral ([1]), both sides of the equality (2) to be proved exist for the indicated parameter values. The left-hand side of (2) satisfies the differential equation

$$\frac{du}{dp} - \frac{1}{2}pu = -\frac{a}{2\sqrt{1+a^2}}\exp\left(-\frac{p^2}{4(1+a^2)}\right)\mathrm{erfc}\left(\frac{p}{2\sqrt{1+a^2}}\right) \tag{3}$$

and tends to 0 as $p \to +\infty$. On the other hand, the unique solution of equation (3) that vanishes as $p \to +\infty$, as is not difficult to see ([2]), is given by the right-hand side of (2), and the theorem is proved.

1. Setting $a = 1$ in (2) and noting at the same time that the integrand in the right-hand side of (2) can be represented in the form

$$\frac{d}{dt}\left[-\sqrt{\frac{\pi}{2}}\operatorname{erfc}^2\left(\frac{t}{2\sqrt{2}}\right)\right],$$

we have

$$\int_0^\infty \exp(-pt - t^2)\operatorname{erf}(t)\,dt = \frac{\sqrt{\pi}}{4}\exp\left(\frac{p^2}{4}\right)\operatorname{erfc}^2\left(\frac{p}{2\sqrt{2}}\right). \tag{4}$$

For $p = 2\sqrt{2z}$, (4) yields a known formula ([1]).

2. Replacing in (2) p by p/a, in the left-hand side t by at, and in the right-hand side t by $2\sqrt{t}$, and resolving the indeterminacy as $a \to i$, we have

$$\int_0^\infty \exp(-pt - a^2t^2)\operatorname{erf}(iat)\,dt = \frac{i\exp(p^2/4a^2)}{2\sqrt{\pi}\,a}\int_{p^2/4a^2}^\infty \frac{\exp(-t)}{t}\,dt. \tag{5}$$

Formula 10.12 ([3]) can be obtained from (5) by multiplying by p and representing i on the right-hand side in the form $-1/i$.

3. Substituting $-p$ for p in (2), subtracting the result thus obtained from (2), and taking into account that $\operatorname{erf}(z)$ is an odd function, we arrive at the equality

$$\int_{-\infty}^\infty \exp(-pt - t^2)\operatorname{erf}(at)\,dt = -\sqrt{\pi}\exp\left(\frac{p^2}{4}\right)\operatorname{erf}\left(\frac{ap}{2\sqrt{1+a^2}}\right). \tag{6}$$

The formula ([4]) for the integral

$$\int_{-\infty}^\infty \exp\left(-(ay+b)^2\right)\operatorname{erf}(y)\,dy$$

can be obtained from (6) by putting $b = p/2$ and replacing t by y/a, and a by $1/a$ ($a \neq 0$).

4. From (2) it follows directly that

$$\int_0^\infty \exp(-pt - t^2)\operatorname{erf}(at)\,dt = 2\sqrt{\pi}\exp\left(\frac{p^2}{4}\right)\left[\frac{1}{4} - \frac{1}{2}\Phi\left(\frac{ap}{\sqrt{2+2a^2}}\right) - T\left(\frac{ap}{\sqrt{2+2a^2}}, \frac{1}{a}\right)\right], \tag{7}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_0^z \exp\left(-\frac{t^2}{2}\right)dt, \qquad T(h,a) = \int_h^\infty \Phi(ax)\,d\Phi(x)$$

are known tabulated functions [5].

Formula (7), as L. N. Bol' shev noted, can also be obtained from probabilistic considerations.

In conclusion we note that an integral of type (1) occurs in solving the problem of the distribution of the maxima of the envelope of the sum of a harmonic signal and quasiharmonic noise with correlation function $\exp(-\alpha\tau^2)\cos\omega_0\tau$ [6].

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Note: Figure translations are in progress. See original paper for figures.