ASYMPTOTIC PROPERTIES OF SOLUTIONS OF ONE CLASS OF INTEGRAL EQUATIONS ARISING IN ELASTICITY THEORY AND MATHEMATICAL PHYSICS

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Abstract

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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF ONE CLASS OF INTEGRAL EQUATIONS ARISING IN ELASTICITY THEORY AND MATHEMATICAL PHYSICS

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The study of the integral equation

$$Kq \equiv \int_{-a}^{a} k(x-\xi)q(\xi)\,d\xi = 2\pi f(x), \qquad |x| \le a, \tag{1}$$

under various assumptions concerning the properties of the kernel $k(t)$ has been the subject of numerous works ([1–7], etc.).

It has been established ([5]) that if $K(z)$–the Fourier transform of the function $k(t)$–is a rational function, then (1), generally speaking, is solvable in closed form. In applications the most frequent cases are those in which $k(t)$ has a logarithmic singularity at zero and decreases exponentially at infinity. In ([3]) a general method is given for constructing an asymptotic solution of equation (1) as $a \to 0$. In the author' s papers ([8,9]) an asymptotic expansion of the solution of equation (1) is constructed under the assumption that $K(z)$ is a meromorphic function. The method of these papers suggests the asymptotic form of the solution of equation (1) for more general kernels arising in mixed problems of elasticity theory and mathematical physics.

In the present note a theorem is presented which answers a number of general questions concerning the properties of the solution of equation (1) for large a. An analytic form of the solution is given and the domain of its existence and uniqueness is indicated. The manner in which the theorem is applied is illustrated by an example.

1. We shall assume that the function $K(z)$ is even, real on the real axis, regular in the domain Ω: $|\sigma| \le \infty$, $|\tau| \le \alpha(\sigma)$; $z = \sigma + i\tau$; $\alpha(\sigma)$ is an even piecewise-smooth function possessing the properties $\alpha(\sigma) \ge \mu > 0$, $\alpha(\sigma) = O(\sigma^{\varepsilon})$, $\sigma \to \infty$, $\varepsilon > 0$. It is assumed that in Ω the function $K(z)$ has no zeros, and that the asymptotic estimate holds

$$K(z) = c^2 z^{-1}[1+O(z^{-1})], \qquad z \in \Omega, \quad |z| \to \infty \qquad \left(k(t) = \int_{-\infty}^{\infty} K(z) e^{izt}\, dz \right). \tag{2}$$

In this case the representation

$$K(z) = K_+(z) K_-(z), \tag{3}$$

is valid, where $K_+(z)$ is regular in the domain $\Omega \cup \operatorname{Im} z > 0$, and $K_-(z)$, respectively, in $\Omega \cup \operatorname{Im} z < 0$, and, moreover,

$$K_+(z) \sim c z^{-0.5}, \qquad z \in \Omega, \quad |z| \to \infty. \tag{4}$$

Let us denote by A the set of functions $\varphi(z)$ regular in the domain $\operatorname{Im} z \le -\delta$ and admitting in $S = \Omega \cap \operatorname{Im} z \le -\delta$ the representation

$$\varphi(z) = \psi(z) z^{-1}, \qquad \max_{z \in S} |\psi(z)| < \infty, \qquad \delta < \mu \tag{5}$$

($\delta > 0$ is an arbitrarily small fixed number).

If in A one introduces the norm by the relation

$$\|\varphi\|_A = \max_{z \in S} |\psi(z)|,$$

then A becomes a Banach space.

Introduce for consideration the operator

$$F(a, z)\varphi \equiv \frac{1}{2\pi i} \int_\Gamma \frac{K_-(t) e^{-2ait} \varphi(t)\, dt}{K_+(t)(t+z)}, \qquad \varphi(t) \in A, \quad \operatorname{Im} z < 0. \tag{6}$$

Here Γ is a contour that is the boundary of the domain Ω in the lower half-plane. It is not difficult to see that $F(a, z)$ acts continuously in A.

By E denote the set of piecewise-smooth contours γ, lying in the domain S, for which the condition is satisfied

$$F(a, z)\varphi \equiv \int_\gamma \frac{K_-(t) e^{-2ait} \varphi(t)\, dt}{K_+(t)(t+z) 2\pi i}, \qquad \varphi(t) \in A, \quad \operatorname{Im} z < 0, \tag{7}$$

and put

$$\inf_{t\in\gamma} |\operatorname{Im} t| = \mu_\gamma, \qquad \gamma \in E. \tag{8}$$

Since in (6) the integrand is regular in S, the contour Γ may be deformed, i.e. E is nonempty.

Theorem. The unique solution in $L_p(-a,a)$ $(p>1)$ of the integral equation (1) with right-hand side $f(x) \in C_1^\lambda(-a,a)$ $(\lambda > 0.5)$ for values $a > a_0$ is given by the relation

$$q(x) = \int_{-\infty}^{\infty} \frac{\Phi(\eta)}{K(\eta)} e^{i\eta x} d\eta - \sum_{k=0}^{\infty} (-1)^k [S(a+x)F^k(a,z)\psi_k +$$

$$+S(a-x)F^k(a,z)\psi_{k+1}]; \tag{9}$$

a_0 is the greatest root of the equation

$$1 = \inf_{\gamma\in E} \max_{z\in\gamma} \frac{e^{-2a_0\mu_\gamma}}{2\pi} \int_\gamma \left| \frac{zK_-(t)e^{-2a_0(it-\mu_\gamma)}}{(z+t)K_+(t)t} \right| |dt|. \tag{10}$$

Moreover, the representation

$$q(x) = \omega(x)(a^2-x^2)^{-0.5}, \qquad \omega(x) \in C(-a,a), \tag{11}$$

is valid, and for $q_n(x)$

$$q_n(x) = \int_{-\infty}^{\infty} \frac{\Phi(\eta)}{K(\eta)} e^{i\eta x} d\eta - \sum_{k=0}^{n} (-1)^k [S(a+x)F^k(a,z)\psi_k +$$

$$+S(a-x)F^k(a,z)\psi_{k+1}]$$

the asymptotic estimate holds

$$[q(x)-q_n(x)](a^2-x^2)^{0.5} = O(\exp[-2a(\mu-\varepsilon)(n+1)]), \qquad x\in[-a,a], \quad a\to\infty; \tag{12}$$

$\varepsilon > 0$ is an arbitrarily small fixed number.

Here the following notation has been introduced:

$$\psi_k(t) = \int_{-\infty}^{\infty} \Phi(\eta)\psi[t,\eta(-1)^k]\, d\eta, \qquad \psi(t,\eta) = \frac{\exp(-i\eta a)}{K_+(\eta)(t+\eta)}, \tag{13}$$

$$S(x)f \equiv \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon} \frac{f(t)e^{-itx}\,dt}{K_+(t)}, \qquad f(x) = \int_{-\infty}^{\infty} \Phi(\eta)e^{i\eta x}d\eta;$$

$$0 < \varepsilon < \mu, \qquad F^k(a,z) \text{ is the } k\text{-th iteration of the operator } F(a,z); \tag{14}$$

$C_1^\lambda(-a,a)$ is the set of functions whose first derivative satisfies a Hölder condition with exponent λ on $[-a,a]$; $C(-a,a)$ is the set of functions continuous on $[-a,a]$.

The proof of the theorem essentially uses the results of the works ([10,11]).

Remark. In relation (10) the contour γ must be chosen in such a way that the sign of the modulus over the function under the integral becomes immaterial.

2. As an example, consider the case occurring in the theory of elasticity ([4]) and mathematical physics ([1,6]), when $k(t) \equiv 2K_0(bt)$ is the Macdonald function. Put $f(x) \equiv \exp i\eta x$, $\operatorname{Im}\eta = 0$. Obviously, $f(x) \in C_1^\lambda(-a,a)$.

For the given case the values $K(z)$, $K_+(z)$, $K_-(z)$ have the form

$$K(z) = (z^2+b^2)^{-0.5}, \qquad K_+(z) = (b-iz)^{-0.5}, \qquad K_-(z) = (b+iz)^{-0.5}. \tag{15}$$

The function $K(z)$ is regular in the complex plane with a cut joining the branch points ib, $-ib$ along the imaginary axis through the point at infinity. Similarly, $K_+(z)$ and $K_-(z)$ are regular in the complex plane with cuts from $-ib$ to $-i\infty$ and from ib to $i\infty$, respectively. The branches of the functions are chosen from the conditions as $z \to \infty$:

$$K(z) \to z^{-1}, \qquad K_+(z) \to z^{-0.5}\exp(i\pi/4), \qquad K_-(z) \to z^{-0.5}\exp(i\pi/4). \tag{16}$$

Obviously, $\Phi(x) = \delta(x-\eta)$, the Dirac delta function, and from (13) it follows that

$$\psi_{2k-1} = e^{i\eta a}\sqrt{b+i\eta}\,(t-\eta)^{-1}, \qquad \psi_{2k} = e^{-i\eta a}\sqrt{b-i\eta}\,(t+\eta)^{-1}. \tag{17}$$

As the contour Γ one may take a contour lying on the left and right banks of the cut joining the points $-ib$, $-i\infty$. The domain S will be the entire lower half-plane with a cut.

Relation (6) for $F(a,z)\varphi$, taking into account the chosen contour Γ, can be represented in the form

$$F(a,z)\varphi = \frac{e^{-2ab}}{\pi i}\int_0^\infty \frac{\sqrt{\tau}\,\varphi[-i(\tau+b)]\,e^{-2a\tau}\,d\tau}{\sqrt{2b+\tau}\,(\tau+b+iz)}. \tag{18}$$

Substituting relation (17) into (18), we obtain

$$F(a,t)\psi_{2k-1} = \frac{e^{i\eta a}\sqrt{b+i\eta}}{\pi}\int_0^\infty \frac{\sqrt{\tau}\,e^{-2a\tau}\,d\tau}{\sqrt{2b+\tau}\,(\tau+b+it)(\tau+b-i\eta)}. \tag{19}$$

The value $F(a,t)\psi_{2k}$ is given by relation (19), in which it is necessary to replace η by $-\eta$.

To construct $F^2(a,z)\psi_k$, it is necessary to apply operation (18) again to (19), and so on.

The solution of the integral equation can be represented in the form

$$\begin{aligned} q(x) = {} & [\operatorname{erf}\sqrt{(b+i\eta)(a+x)} + \operatorname{erf}\sqrt{(b-i\eta)(a-x)} - 1]\,K^{-1}(\eta)\exp i\eta x \\ & + \frac{\sqrt{b-i\eta}}{\sqrt{\pi(a+x)}}\exp[-b(a+x)-i\eta a] + \frac{\sqrt{b+i\eta}}{\sqrt{\pi(a-x)}}\exp[-b(a-x)+i\eta a] \\ & - \sum_{k=1}^{\infty}(-1)^k\left[S(a+x)F^k(a,z)\psi_k + S(a-x)F^k(a,z)\psi_{k+1}\right], \qquad \operatorname{erf} x = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}\,dt. \end{aligned}$$

The expression standing outside the summation sign represents the zero term of the asymptotic expansion of the solution as $a \to \infty$. It was first obtained by V. M. Aleksandrov ([4]).

From property (12) it follows that the order of the remainder term is $O(\exp[2a(b-\varepsilon)])$ $(a \to \infty)$. This circumstance also explains the high effectiveness of the zero term of the asymptotics in contact problems.

To determine a_0 it is necessary to solve equation (10). Let us find an approximate value $a^* > a_0$ by solving equation (10) for the case when Γ is taken as the contour γ. The equation for a^* takes the form

$$1 = \max_{z\in[0,\infty]} \frac{e^{-2a^*b}}{\pi}\int_0^\infty \frac{\sqrt{\tau}\,ze^{-2a^*\tau}\,d\tau}{\sqrt{2b+\tau}\,|\tau+b+iz|(\tau+b)}. \tag{20}$$

From (20) one obtains an equation for $a_1 > a^*$ of the form

$$\pi = -\operatorname{Ei}(-2a_1b), \qquad \operatorname{Ei}(-x) = -\int_x^\infty \frac{e^{-t}}{t}\,dt. \tag{21}$$

With the aid of the tables [12], the following estimate for a_1 is determined:

$$a_1 < 0.015b^{-1}. \tag{22}$$

Thus, the series (9) represents the solution of the integral equation (1) for $0.015b^{-1} \leq a \leq \infty$.

In contact problems of the theory of elasticity [4] the value is $b = 2$, and the series indicated above represents the solution on the interval

$$0.0075 \leq a \leq \infty. \tag{23}$$

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Note: Figure translations are in progress. See original paper for figures.