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Abstract

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THEORY OF ELASTICITY

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ON THE INTEGRATION OF A SYSTEM OF EQUATIONS OF ELASTIC EQUILIBRIUM OF A PLATE

The problem of studying the elastic equilibrium of a plate, following the shell theory developed in ([1,2]), leads to the following elliptic system of equations of the 12th order (approximation of order $N = 1$):

$$\begin{gathered}\mu\Delta u_1 + (\lambda+\mu)\partial\theta_1/\partial x + \lambda\partial v/\partial x = X_1,\\ \mu\Delta u_2 + (\lambda+\mu)\partial\theta_1/\partial y + \lambda\partial v/\partial y = X_2 \quad (\theta_1 = \partial u_1/\partial x + \partial u_2/\partial y);\\ \mu\Delta v - 12((\lambda+2\mu)v + \lambda\theta_1) = X_3;\end{gathered} \tag{1}$$

$$\begin{gathered}\mu\Delta v_1 + (\lambda+\mu)\partial\theta_2/\partial x - 12\mu(\partial u/\partial x + v_1) = Y_1,\\ \mu\Delta v_2 + (\lambda+\mu)\partial\theta_2/\partial y - 12\mu(\partial u/\partial y + v_2) = Y_2,\\ (\theta_2 = \partial v_1/\partial x + \partial v_2/\partial y);\\ \mu\Delta u + \mu\theta_2 = Y_3,\end{gathered} \tag{2}$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are prescribed functions; x and y are dimensionless (Descartes) coordinates; the metric quadratic form on the middle plane has the form $ds^2 = 4h^2(dx^2+dy^2)$, where $2h$ is the thickness of the plate (h is constant).

The functions entering into (1) and (2) have the following kinematic meaning: $u+tv$ is the normal deflection, u_1+tv_1 and u_2+tv_2 are tangential displacements on the plane parallel to the middle plane and at a distance $2ht$ from it, $-1/2 \leq t \leq 1/2$; u is the deflection of the middle plane, v is the elongation of transverse fibers. In the classical theory of shells, based on the Kirchhoff-Love hypothesis, the quantity v is neglected, taking $v = 0$.

For $X_i = Y_i = 0$ $(i = 1, 2, 3)$, systems (1) and (2) are integrated in explicit form ([1-3])

$$v = \chi - \frac{\lambda}{\lambda+2\mu}\Delta\varphi; \tag{3}$$

$$u_1 = -\frac{\lambda}{48(\lambda+\mu)}\frac{\partial\chi}{\partial x} + \frac{\partial\varphi}{\partial x} - \frac{\partial\varphi^*}{\partial y}, \qquad u_2 = -\frac{\lambda}{48(\lambda+\mu)}\frac{\partial\chi}{\partial y} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi^*}{\partial x}; \quad (4)$$

$$u = -\omega + B_0\Delta\omega \quad (B_0 = (\lambda+2\mu)/12\mu); \qquad (5)$$

$$v_1 = \partial\omega/\partial x - \partial\psi/\partial y, \qquad v_2 = \partial\omega/\partial y + \partial\psi/\partial x, \qquad (6)$$

where χ and ψ are arbitrary solutions of the equations*

$$\Delta\chi - k^2\chi = 0, \qquad \Delta\psi - m^2\psi = 0 \quad (k^2 = 48(\lambda+\mu)/(\lambda+2\mu),\ m^2 = 12), \quad (7)$$

$\varphi, \varphi^*, \omega$ are biharmonic functions, with φ and ω arbitrary, while φ^* is expressed in terms of φ; if, for example, φ is written in the form

$$\varphi = \frac{\lambda+2\mu}{2(3\lambda+2\mu)}\left(z\bar{f} + \bar{z}f - \frac{1}{2}(f_0 + \bar{f}_0)\right), \qquad (8)$$

then

$$\varphi^* = i\,\frac{2(\lambda+\mu)}{3\lambda+3\mu}(z\bar{f} - \bar{z}f). \qquad (9)$$

Here f and f_0 are arbitrary analytic functions of $z = x + iy$.

* The equation $\Delta\psi - m^2\psi = 0$ first occurs in E. Reissner (^{6}), but there $m^2 = 10$. The reason for this discrepancy is indicated in (^{3}).

Let $\widetilde{X}_i, \widetilde{Y}_i$ $(i = 1, 2, 3)$ be functions satisfying the equations

$$\Delta\Delta\widetilde{X}_i = X_i, \qquad \Delta\Delta\widetilde{Y}_i = Y_i \qquad (i = 1, 2, 3). \qquad (10)$$

Differentiating the first equation of system (1) with respect to x, the second with respect to y, and then adding the results, by virtue of (10) we shall have

$$(\lambda+2\mu)\Delta\theta_1 + \lambda\Delta v = \Delta\Delta\nabla^\alpha\widetilde{X}_\alpha, \qquad (11)$$

where ∇^α denotes the symbol of the contravariant derivative.

Equation (11) is satisfied by the function

$$(\lambda+2\mu)\theta_1 + \lambda v = \Delta\nabla^\alpha\widetilde{X}_\alpha. \qquad (12)$$

Hence we have

$$\theta_1 = -\frac{\lambda}{\lambda+2\mu}v + \frac{1}{\lambda+2\mu}\Delta\nabla^\alpha \widetilde{X}_\alpha. \tag{13}$$

Substituting this into the third equation of system (1), we obtain

$$(\Delta - k^2)v = \Delta\left(\frac{1}{\mu}\Delta\widetilde{X}_3 + \frac{12\lambda}{\mu(\lambda+2\mu)}\nabla^\alpha \widetilde{X}_\alpha\right). \tag{14}$$

Let $L_k(g)$ be an operator giving some particular solution of the equation $(\Delta - k^2)w = g$ ($k^2 = \mathrm{const}$). The operator L_k can always be chosen in such a form that the permutation formulas (in the case of Cartesian coordinates) $L_k(\nabla_\alpha g) = \nabla_\alpha L_k(g)$ $(\alpha = 1, 2)$ hold.

Representing v by the formula

$$v = \Delta L_k\left(\frac{1}{\mu}\Delta\widetilde{X}_3 + \frac{12\lambda}{\mu(\lambda+2\mu)}\nabla^\alpha \widetilde{X}_\alpha\right) \tag{15}$$

and inserting this expression into (13), we obtain

$$\theta_1 = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = \Delta\left(\frac{\lambda}{\lambda+2\mu}\nabla^\alpha \widetilde{X}_\alpha - \frac{\lambda}{\mu(\lambda+2\mu)}L_k\left(\Delta\widetilde{X}_3 + \frac{12\lambda}{\lambda+2\mu}\nabla^\alpha \widetilde{X}_\alpha\right)\right). \tag{16}$$

Differentiating now the first equation of system (1) with respect to y, the second with respect to x, and then subtracting, we shall have

$$\mu\Delta\theta_1^* = \Delta\Delta c^{\alpha\beta}\nabla_\beta \widetilde{X}_\alpha, \qquad \theta_1^* = \partial u_1/\partial y - \partial u_2/\partial x, \tag{17}$$

where $c^{\alpha\beta}$ is the contravariant discriminant tensor (4). From (17) we have

$$\theta_1^* = \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x} = \nabla\left(\frac{1}{\mu}c^{\alpha\beta}\Delta_\beta \widetilde{X}_\alpha\right). \tag{18}$$

Equalities (16) and (18) can be written in the following complex form:

$$2\partial(u_1 - iu_2)/\partial\bar{z} = \theta_1 + i\theta_1^*$$

$$= \Delta\left(\frac{1}{\lambda+2\mu}\nabla^\alpha \widetilde{X}_\alpha - \frac{\lambda}{\mu(\lambda+2\mu)}L_k\left(\Delta\widetilde{X}_3 + \frac{12\lambda}{\lambda+2\mu}\nabla^\alpha \widetilde{X}_\alpha\right) + \frac{i}{\mu}c^{\alpha\beta}\nabla_\beta \widetilde{X}_\alpha\right). \tag{19}$$

Since $\Delta = 4\partial^2/\partial z\bar{z}$, from (19) we have

$$u_1+iu_2 = 2\frac{\partial}{\partial z}\left(\frac{1}{\lambda+2\mu}\nabla^\alpha\widetilde{X}_\alpha - \frac{\lambda}{\mu(\lambda+2\mu)}L_k\left(\Delta\widetilde{X}_3+\frac{12\lambda}{\lambda+2\mu}\nabla^\alpha\widetilde{X}_\alpha\right) - \frac{i}{\mu}c^{\alpha\beta}\nabla_\beta\widetilde{X}_\alpha\right). \tag{20}$$

Formulas (15) and (20) give particular solutions of the nonhomogeneous system (1).

Let X_1, X_2 be the covariant components of a vector, and X_3 a scalar.

Representing now the Laplace operator in the form $\Delta = \nabla^\alpha\nabla_\alpha = \nabla^2$, formulas (15) and (20) can be written in tensor form:

$$v = \nabla^2 L_k\left(\frac{1}{\mu}\nabla^2\widetilde{X}_3 + \frac{12\lambda}{\mu(\lambda+2\mu)}\nabla^\alpha\widetilde{X}_\alpha\right), \tag{21}$$

$$u_\alpha = \nabla^\beta\left(a_{\alpha\beta}\left(\frac{1}{\lambda+2\mu}\nabla^\gamma\widetilde{X}_\alpha - \frac{\lambda^\gamma}{\mu(\lambda+2\mu)}L_k\left(\nabla^2\widetilde{X}_3+\frac{12\lambda}{\lambda+2\mu}\nabla^\gamma\widetilde{X}_\gamma\right)\right) + \frac{1}{\mu}c_{\alpha\beta}c^{\gamma\nu}\nabla_\nu\widetilde{X}_\gamma\right), \tag{22}$$

where $a_{\alpha\beta}$ is the covariant metric tensor, and $c_{\alpha\beta}$ is the covariant discriminant tensor.

With respect to arbitrarily chosen curvilinear coordinates x^1, x^2, equation (1) can be written in the form

$$\mu\nabla^\alpha\nabla_\alpha u_\beta + (\lambda+\mu)\nabla_\beta\nabla^\alpha u_\alpha + \lambda\nabla_\beta v = X_\beta \quad (\beta = 1,2),$$

$$\mu\nabla^2 v - 12((\lambda+2\mu)v + \lambda\nabla^\alpha u_\alpha) = X_3. \tag{23}$$

Let us now turn to equations (2). Differentiating the first equation with respect to x, the second with respect to y, and then adding the results, by virtue of the third equation we obtain

$$(\lambda+2\mu)\Delta\theta_2 = 12\Delta\Delta\widetilde{Y}_3 + \Delta\Delta\nabla^\alpha\widetilde{Y}_\alpha.$$

Hence we have

$$\theta_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \Delta\left(\frac{12}{\lambda+2\mu}\widetilde{Y}_3 + \frac{1}{\lambda+2\mu}\nabla^\alpha\widetilde{Y}_\alpha\right). \tag{24}$$

Differentiating the first equation of system (2) with respect to y, the second with respect to x, and then subtracting, we shall have

$$\mu(\Delta\theta_2^* - m^2\theta_2^*) = \Delta\Delta c^{\alpha\beta}\nabla_\beta \widetilde{Y}_\alpha \quad (\theta_2^* = \partial v_1/\partial y - \partial v_2/\partial x;\ m^2 = 12).$$

Hence we have

$$\theta_2^* = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} = \Delta\Delta\left(\frac{1}{\mu}L_m\left(c^{\alpha\beta}\nabla_\beta \widetilde{Y}_\alpha\right)\right). \tag{25}$$

Equalities (24) and (25) can be written in complex form as follows:

$$2\frac{\partial(v_1 - iv_2)}{\partial\bar{z}} = \theta_2 + i\theta_2^* = \Delta\left(\frac{12}{\lambda+2\mu}\widetilde{Y}_3 + \frac{1}{\lambda+2\mu}\nabla^\alpha \widetilde{Y}_\alpha + \frac{i}{\mu}\Delta L_m\left(c^{\alpha\beta}\nabla_\beta \widetilde{Y}_\alpha\right)\right),$$

i.e., we may assume that

$$v_1 + iv_2 = 2\frac{\partial}{\partial\bar{z}}\left(\frac{12}{\lambda+2\mu}\widetilde{Y}_3 + \frac{1}{\lambda+2\mu}\nabla^\alpha \widetilde{Y}_\alpha - \frac{i}{\mu}\Delta L_m\left(c^{\alpha\beta}\nabla_\beta \widetilde{Y}_\alpha\right)\right) \tag{26}$$

or, in tensor form,

$$v_\alpha = \nabla^\beta\left(a_{\alpha\beta}\left(\frac{12}{\lambda+2\mu}\widetilde{Y}_3 + \frac{1}{\lambda+2\mu}\nabla^\gamma \widetilde{Y}_\gamma\right) + \frac{1}{\mu}c_{\alpha\beta}\nabla^2 L_m\left(c^{\gamma\nu}\nabla_\nu \widetilde{Y}_\gamma\right)\right) \quad (\alpha = 1, 2). \tag{27}$$

Now, by virtue of the third equation of system (2) and equality (24), for u we obtain the formula

$$u = \frac{1}{\mu}\nabla^2 \widetilde{Y}_3 - \frac{12}{\lambda+2\mu}Y_3 - \frac{1}{\lambda+2\mu}\nabla^\alpha \widetilde{Y}_\alpha. \tag{28}$$

Formulas (27) and (28) give particular solutions of the system of equations (2). This system in tensor form can be written as

$$\mu\nabla^\alpha\nabla_\alpha v_\beta + (\lambda+\mu)\nabla_\beta\nabla^\alpha v_\alpha - 12\mu(\nabla_\beta u + v_\beta) = Y_\beta \quad (\beta = 1, 2),$$

$$\mu\nabla^2 u + \mu\nabla^\alpha v_\alpha = Y_3. \tag{29}$$

Now let us return to the question of constructing particular solutions of the equations $\Delta\Delta w = g$ and $(\Delta - k^2)w = g$ ($k^2 = \text{const}$). Their particular solutions can be constructed, respectively, by the formulas

$$w = P_0(g) = \frac{1}{8\pi}\iint_E |z-\xi|^2 \ln|z-\xi|\, g(\xi,\eta)\, d\xi\, d\eta; \tag{30}$$

$$w = L_k(g) = \frac{1}{2\pi} \iint_E K_0(k,r)\, g(\xi,\eta)\, d\xi\, d\eta, \qquad r = |z-\xi|, \tag{31}$$

where K_0 is the modified Bessel function of the second kind [5]. If g has continuous partial derivatives on the plane E, then it is easy to verify the validity of the following commutation formulas (with respect to Cartesian coordinates):

$$P_0(\nabla_\alpha g) = \nabla_\alpha P_0(g), \qquad L_k(\nabla_\alpha g) = \nabla_\alpha L_k(g) \quad (\alpha = 1,2).$$

Lemma. *Let g_1, g_2 be the covariant components of a vector. Suppose that they are continuously differentiable in E and vanish outside a circle of sufficiently large radius. Then there exist scalars u, v, by means of which g_1, g_2 are expressed in the form $g_\alpha = \nabla^\beta(a_{\alpha\beta}u + c_{\beta\alpha}v)$ $(\alpha = 1,2)$.*

As u and v one may take scalars satisfying the equations: $\nabla^2 u = \nabla^\alpha g_\alpha$, $\nabla^2 v = c^{\alpha\beta}\nabla_\beta g_\alpha$; they may be expressed by the formulas

$$u = \frac{1}{2\pi} \iint_E \ln|z-\xi|\, \nabla^\alpha g_\alpha\, d\xi\, d\eta, \qquad v = \frac{1}{2\pi} \iint_E \ln|z-\xi|\, c^{\alpha\beta}\nabla_\beta g_\alpha\, d\xi\, d\eta.$$

On the basis of this lemma it is easily proved that particular solutions of the equations $\nabla^2 w_\alpha = g_\alpha$, $(\nabla^2 - k^2)w_\alpha = g_\alpha$ can be constructed, respectively, by the formulas

$$w_\alpha = \nabla^\beta(a_{\alpha\beta}P_0(u) + c_{\beta\alpha}P_0(v)), \qquad w_\alpha = \nabla^\beta(a_{\alpha\beta}L_k(u) + c_{\beta\alpha}L_k(v)).$$

Obviously, w_1, w_2 are the components of a covariant vector.

Let g satisfy the equation $\nabla^2 g - p^2 g = 0$, $p^2 = \text{const} \neq 0$. Then $P_0(g)$ and $L_k(g)$ can be expressed by the formulas

$$P_0(g) = \frac{1}{p^4} g, \qquad L_k(g) = \begin{cases} \dfrac{1}{p^2 - k^2}\, g, & \text{if } p^2 \neq k^2, \\ r\,\dfrac{\partial g}{\partial r}, & \text{if } p^2 = k^2. \end{cases}$$

If g satisfies the equation $\nabla^{2n} g = 0$, then

$$L_p(g) = -\frac{1}{p^4}\left(g + \frac{1}{p^2}\nabla^2 g + \cdots + \frac{1}{p^{2n-2}}\nabla^{2n-2} g\right),$$

$$g = \sum_{k=0}^{n-1} \left(z^k \bar{f}_k + \bar{z}^k f_k\right),$$

$$w = P_0(g) = \frac{1}{16} \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} \left(z^{k+2} \bar{f}_k + \bar{z}^{k+2} f_k \right),$$

where f_k are arbitrary analytic functions of z.

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Note: Figure translations are in progress. See original paper for figures.