GAUSSIAN MEASURES, CAUCHY MEASURES, AND \(\varepsilon\)-ENTROPY

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Abstract

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MATHEMATICS

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GAUSSIAN MEASURES, CAUCHY MEASURES, AND ε-ENTROPY

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Let x_t, $t \in T$, be a Gaussian random process with zero mean and correlation function $r(s,t)$. Let E be some Banach space of functions on T. Below we give conditions which can often be checked from the function $r(s,t)$ and which in most cases make it possible to decide whether the realizations of the process belong to the space E.

1. The correlation function $r(s,t)$ determines an embedding $T \subset \mathcal{H}$ of the set T into the Hilbert space $\mathcal{H}$ of all measurable linear functionals over the Gaussian process ([1]). In what follows the separability of $\mathcal{H}$ is always assumed. Consider the linear hull $\mathcal{L}(T)$ and on it the norm induced by the duality $(E, \mathcal{L}(T))$; let $K_1 \subset \mathcal{L}(T)$ be the unit ball. If E has full measure, then K_1 is relatively compact in $\mathcal{H}$ and the embedding $K_1 \subset \mathcal{H}$ is continuous from $\sigma(\mathcal{L}(T), E)$. Let $K = \overline{K}_1$; $\mathcal{L}(K) \subset \mathcal{H}$ is Banach and consists of all linear forms continuous on E and measurable with respect to the σ-algebra generated by the functionals from T. In what follows the compact topology is considered on K.

Proposition 1. E consists of linear forms bounded on K, and contains the set of all linear forms continuous on K.

Proposition 2. Let L be a linear space with Gaussian measure μ, and let $L_1 \subset L$ be a measurable linear subspace. Then either $\mu L_1 = 0$, or $\mu L_1 = 1$ (see ([2])).

The process x_t is stochastically continuous with respect to the metric on T induced from $\mathcal{H}$, and therefore its realizations are measurable. The set K is restored from $T \subset \mathcal{H}$. In particular, if E is a space of functions with the uniform norm, then K is the closed convex hull of T. In any case, we arrive at the following general scheme. A Banach space E and another normed space E_1 in duality with E are given. On E a separable Gaussian weak distribution (a generalized random process) is given, i.e., a mapping (which we may regard as an embedding) of E_1 into the subspace $\mathcal{H}$ of Gaussian-distributed functions of the space S_μ of all measurable functions on some set with a separable measure μ. This embedding is bounded and continuous from the topology $\sigma(E_1, E)$, and the unit sphere $K_1 \subset E_1$ is compact in $\mathcal{H}$. The problem is to find, in terms of $K_1 \subset \mathcal{H}$, conditions for the extendability of the weak distribution to a measure in E.

2. Conditions for the possibility of extending a weak Gaussian distribution to a measure, close to necessary and sufficient ones, can be given in terms of the ε-entropy of the compact set $K \subset \mathcal{H}$. The entropy type (see, for example, ([3])) of the compact set K is the number

$$\rho(K) = \limsup_{\varepsilon \to 0} \frac{\log H(K; V_{\mathcal{H}}, \varepsilon)}{\log 1/\varepsilon},$$

where $H(K; V_{\mathcal{H}}, \varepsilon) = \log N(K; V_{\mathcal{H}}, \varepsilon)$ is the ε-entropy of K, and $V_{\mathcal{H}}$ is the unit ball of $\mathcal{H}$. Let $K = \overline{K}_1$.

Theorem 1. If $\rho(K) > 2$, then the realizations of the process x_t, $t \in K$, are unbounded with probability 1.

Theorem 2. If $\rho(K) < 2$, then the realizations of the process x_t, $t \in K$, are bounded with probability 1.

Consider the weak Cauchy distribution whose characteristic functional on an element $h \in \mathcal{H}$ is equal to $\exp(-\|h\|_{\mathcal{H}})$. Since

$$\exp(-r) = \int_0^\infty \exp\left(-\frac{1}{2}(r\sigma)^2\right) \left(\sqrt{\frac{2}{\pi}} \frac{1}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\right)\right) d\sigma, \tag{1}$$

and $\exp(-\frac{1}{2}\|h\|_{\mathcal{H}})$ is the characteristic functional of our Gaussian distribution, the stock of linearly measurable sets of full measure for the Cauchy measure $\varkappa$ and the Gaussian measure μ is one and the same. The conditional measures [4] of the measure under a (measurable) partition of the linear space with Gaussian measure μ into rays are δ-measures on almost every ray (cf. Shneiberg's theorem [5]), and formula (1) shows that for the Cauchy measure $\varkappa$ the conditional measures have density

$$p(\sigma) = \sqrt{\frac{2}{\pi}} \frac{1}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\right).$$

For any measurable A,

$$\varkappa A = \int_0^\infty \mu^\sigma A \, p(\sigma) \, d\sigma,$$

where μ^σ has characteristic functional $\exp\left(-\frac{1}{2}(\|h\|\sigma)^2\right)$.

Proposition 3 (Shleffli-Slepian theorem [6,7]). *Let $A, B \subset \mathcal{H}$ and suppose there exists a mapping ψ of the set A onto B such that for $f, g \in A$*

$$\frac{(f,g)+1}{(\|f\|^2+1)^{1/2}(\|g\|^2+1)^{1/2}} \le \frac{(\psi f,\psi g)+1}{(\|\psi f\|^2+1)^{1/2}(\|\psi g\|^2+1)^{1/2}}.$$

Then $\varkappa A^0 \le \varkappa B^0$, where C^0 is the set of linear forms from the space with measure that do not exceed C in absolute value on C.

Choose an orthonormal basis $\{e_k\}$ in $\mathcal{H}$. By a parallelepiped $\pi(\{a_k\})$ with parameters $a_k \downarrow 0$ we shall mean the set

$$\pi = \{h : \sup a_k^{-1}|(h,e_k)| \le 1\}.$$

Let $\hat{\pi}$ be the set of vertices of π. A GB-set is [8] a set $A \subset \mathcal{H}$ for which $\mu(\lambda A^0) > 0$ for some $\lambda > 0$.

Proposition 4. *In the class of parallelepipeds the condition $\rho(\hat{\pi}) \le 2$ is necessary for the GB-property. Indeed, it can be shown (cf. [10]) that $H(\varepsilon) = H(\hat{\pi}(\{a_k\}); V_{\mathcal{H}}\varepsilon)$ for small ε is equivalent to the number of terms of the sequence*

$$\left\{ \left(\sum_{k=n}^{\infty} a_k^2 \right)^{1/2} \right\}$$

greater than ε. Therefore [9] the quantity

$$\rho(\hat{\pi}) = \limsup \left(\frac{\log H(\varepsilon)}{\log 1/\varepsilon} \right)$$

is equal to the exponent of convergence of this sequence:

$$\rho(\hat{\pi}) = \inf \left\{ \alpha : \sum_n \left[\left(\sum_{k=n}^{\infty} a_k^2 \right)^{1/2} \right]^{\alpha} < \infty \right\}.$$

If now $\rho(\hat{\pi}) > 2$, then $\sum n a_n^2 = \infty$, and hence $\sum a_n = \infty$, i.e. π is not a GB-set (the three-series theorem).

Proof of Theorem 1. Let $\{h_j^{(k)}\}, j = 1, \dots, M_k$, be a sequence of ε_k-nets K, for which

$$\lim \frac{\log\log M_k}{\log 1/\varepsilon} > 2.$$

For some $c = c(K)$, by Proposition 3 we obtain

$$\varkappa K^0 \leq \varkappa\{c\varepsilon_k e_1, \dots, c\varepsilon_k e_{M_k}\}^0.$$

A direct calculation verifies that

$$\mu^\sigma\{c\varepsilon_k e_1, \dots, c\varepsilon_k e_{M_k}\}^0 \to 0$$

for any $\sigma > 0$, and then also

$$\varkappa\{c\varepsilon_k e_1, \dots, c\varepsilon_k e_{M_k}\}^0 \to 0.$$

Since $\varkappa K^0 = 0$, it follows that also $\varkappa\mathcal{L}(K^0) = 0$ and $\mu\mathcal{L}(K^0) = 0$.

Proof of Theorem 2. Let $\rho(K) < 2$. Consider a sequence $\{h_j^{(k)}\}, j = 1, \dots, N(2^{-k})$, of minimal 2^{-k}-nets of K, and let $N_{k_0} = N(2^{-k_0}) = 1$, while $N_{k_0+1} > 1$. Let

$$\bar{H}_k = -[-\log N_k].$$

Construct a parallelepiped $\pi = \pi(\{a_k\})$, for which the first $\bar{H}_{k_0+1}$ terms are equal to $2^{-k}c$, the next $\bar{H}_{k_0+2}$ terms are each equal to $2^{-k-1}c$, etc. For this parallelepiped $\rho(\hat{\pi}) = \rho(K)$, and there will be found such $c = c(K)$ and such a subset of the set π that for it there exists a "contracting," in the sense of Proposition 3, mapping onto the union of all the chosen ε-nets. Thus

$$\varkappa K^0 \geq \varkappa\pi^0 > 0,$$

i.e., indeed,

$$\mu\mathcal{L}(K^0) = 1.$$

3. **Proposition 5.** If Q is a set measurable with respect to the point O and

$$q = \int_0^1 \sqrt{\frac{2}{\pi}}\, \frac{1}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2}\right) d\sigma \approx 0.3174,$$

then

$$\mu Q \geq (\varkappa Q - q)/(1 - q).$$

The proof follows from the formula for conditional Cauchy measures.

Corollary (Dudley' s theorem [8]**).** If $\rho(K) < 2$, then the space $E_0(K)$ of linear forms continuous on K has full measure.

Indeed,

$$E_0(K) = \bigcap_{\lambda>0} \lambda \bigcup_n (K \cap L_n)^0,$$

where L_n is a decreasing sequence of closed subspaces with zero intersection. For any λ, if n is sufficiently large, then $\varkappa(\lambda(K \cap L_n)^0)$, as follows from the proof of Theorem 2, is sufficiently close to 1, and then, by Proposition 4, also $\mu(\lambda(K \cap L_n))^0$.

Proposition 6. If the measure of the space $E_0(K)$ of continuous forms is zero, then for some $\varepsilon > 0$ the realizations with probability 1 are not bounded by the number ε.

Indeed, by the zero-one law, $\mu\left(\lambda \bigcup_n (K \cap L_n)^0\right)$ is then equal to zero for some $\lambda > 0$.

Let us summarize the conclusion from Theorems 1 and 2, Proposition 1, and the corollary:

Theorem 3. Suppose that in a Banach space E a weak Gaussian distribution is given by means of a set of linear functionals E_1, with a characteristic functional $\chi(x) = \exp\left(-\frac{1}{2}A(x,x)\right)$ that is continuous in norm on E_1. In order that this weak distribution extend to a measure, it is necessary that the unit ball $K_1 \subset E_1$, in the sense of duality (E, E_1), have, in the norm $\|x\|^2_{\mathcal{H}} = A(x,x)$, entropy type $\rho(K_1) \leq 2$, and it is sufficient that it have entropy type $\rho(K_1) < 2$.

With the aid of this theorem one can obtain, for example, the well-known Hunt-Belyaev conditions ([11]) for continuity of realizations of a stationary Gaussian process.

4. A necessary and sufficient condition for an ellipsoid K to possess the GB property is the condition

$$\int_0^1 \varepsilon^2 dH(\varepsilon) > -\infty.$$

It seems implausible that, in terms of ε-entropy, there should exist an exhaustive answer in the general case. Let, for example, $\pi = \pi(\{1/k \ln k\})$.

Then $\pi \notin GB$, but

$$\int_0^1 \varepsilon^2 dH(\pi; V_{\mathcal{H}}, \varepsilon) > -\infty,$$

therefore, for an ellipsoid with the same growth H (there exists an ellipsoid K for which $H_\pi(a\varepsilon) \leq H_k(\varepsilon) \leq H_\pi(b\varepsilon)$) the GB property is satisfied. Passing to a sequence of finite-dimensional sections also does not improve the situation.

5. The results of items 2 (Theorems 1 and 2) and 4 were reported by the author at the International Congress of Mathematicians in 1966. Recently Dudley' s paper ([8]) appeared, in which he notes that Theorem 2 had also been proved by Strassen (unpublished), and he himself finds a certain strengthening of this theorem, which can also be obtained by our methods (a corollary of Proposition 5). The notation for the GB property belongs to him as well ([8]).

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Note: Figure translations are in progress. See original paper for figures.