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MATHEMATICS

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Abstract

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MATHEMATICS

V. Yu. SHELEPOV

ON THE INDEX OF AN INTEGRAL OPERATOR OF POTENTIAL TYPE IN THE SPACE L_p

(Presented by Academician I. N. Vekua on 14 XI 1968)

1. In the work (1), Ya. B. Lopatinskii studied integral equations of the form

$$(I-G)u(x) \equiv u(x) - \int_\Gamma \frac{1}{|x-y|} G\left(x, y, \nu(x), \nu(y), \frac{x-y}{|x-y|}\right) u(y)\, d_{yS} = f(x). \tag{1}$$

to which (2) boundary-value problems for linear elliptic systems of differential equations are reduced. Here Γ is a closed rectifiable Jordan curve in the plane; $x, y \in \Gamma$; $\nu(x)$ is the inward normal to Γ; $G(\ )$ is a matrix of size $m \times m$; $u(y)$ is a column of unknown functions.

Considering (1) in the space L_1 with a certain weight, he obtained a formula for the index of this equation under the assumption that Γ consists of a finite number of Lyapunov arcs and has no return points. E. N. Parasyuk generalized (3) this result to the case of curves of bounded rotation without return points. In doing so, he also considered equation (1) in the space L_1 with a certain specially chosen weight depending on Γ.

The aim of the present work is, following the scheme of (1), to study equation (1) in the function spaces L_p, $1 < p < \infty$. We shall show that in this case its index depends both on p and on the geometry of the curve.

2. Let Γ be a closed Jordan curve of bounded rotation without return points. This means (4) that the angle $\theta(s)$ between the positive tangent to Γ and the axis of abscissas, referred to the length of the arc, is on the interval $[0, S]$ a function of bounded variation, all of whose jumps are, in absolute value, less than π. We shall consider equation (1) in the space $L_p = L_p^{(m)}$ of functional m-columns $u(x)$ with elements u_i and norm

$$\|u\| = \sum_{i=1}^{m} \|u_i\|_{L_p}.$$

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With respect to the elements G_{ij} of the matrix $G(x, y, \xi, \eta, \zeta)$, defined for $x, y \in \Gamma$, $|\xi| = |\eta| = |\zeta| = 1$, we shall assume that they: 1) satisfy a Lipschitz condition in all arguments, and 2)

$$G_{ij}(x, y, \xi, \eta, \zeta) = O\left(|(\xi, \zeta)| + |(\eta, \zeta)|\right).$$

As was indicated in ([5]), the following holds.

Theorem 1. *The operator G is bounded, and, in the absence of corner points on Γ, is a completely continuous operator in the space L_p, $1 < p < \infty$.*

Proof (in the case of a nonclosed curve) follows from the estimate

$$\frac{1}{|x-y|}\,|G_{ij}(\ )| < \frac{\theta_1(s)-\theta_1(\sigma)}{s-\sigma} + \frac{\theta_2(s)-\theta_2(\sigma)}{s-\sigma},$$

which holds by virtue of property 2). Here

$$\theta_1(s) = \bigvee_0^s(\theta), \qquad \theta_2(s) = \theta_1(s) - \theta(s).$$

In the case of an open curve the estimate holds:

$$\|G\| \leq M \sqrt[s]{}(\theta), \qquad M = \text{const}.$$

It, together with condition 1), makes it possible, by successively simplifying the operator $I - G$ as is done in ([1]), to reduce it to an operator of convolution type on the half-axis. The results of I. Ts. Gokhberg and M. G. Krein ([6]) are then applied to the latter.

3. Introduce the following notation: $a_1, a_2, \ldots$ is the set of all corner points of the curve Γ; $0 < \omega_k < 2\pi$ is the interior angle at the point a_k; τ_k is the unit vector of the left positive tangent to Γ at the point a_k; ν_k is the corresponding inner normal;

$$\xi_k(t) = \frac{(e^{-t/2}\cos\omega_k - e^{t/2})\tau_k - \nu_k e^{-t/2}\sin\omega_k}{\sqrt{e^t + e^{-t} - 2\cos\omega_k}};$$

$$H^{(1)}_{\omega_k}(z) = \int_{-\infty}^{\infty} \frac{e^{i(z-i/2)t}}{\sqrt{e^t + e^{-t} - 2\cos\omega_k}}\, G(a_k,\ a_k,\ -\tau_k\sin\omega_k - \nu_k\cos\omega_k,\ \nu_k, -\xi_k(t))\, dt;$$

$$H^{(2)}_{\omega_k}(z) = \int_{-\infty}^{\infty} \frac{e^{i(z-i/2)t}}{\sqrt{e^t + e^{-t} - 2\cos\omega_k}}\, G(a_k,\ a_k,\ \nu_k,\ -\tau_k\sin\omega_k - \nu_k\cos\omega_k,\ \xi_k(-t))\, dt;$$

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$$\Delta_{\omega_k}(z) = \det\{I - H^{(1)}_{\omega_k}(z) H^{(2)}_{\omega_k}(z)\}.$$

The last function, analytic in the strip $0 < \operatorname{Im} z < 1$, has there no more than a countable set of zeros.

Theorem 2. If $1 < p < \infty$ is such that the functions $\Delta_{\omega_k}(z)$, $k = 1, 2, \ldots$, do not vanish on the line $\operatorname{Im} z = 1/p$, then equation (1) is normally solvable in the space L_p, and its index is equal to

$$\varkappa_p = -\frac{1}{2\pi} \sum_{k=1}^{\infty} \int_{-\infty + i/p}^{\infty - i/p} d \arg \Delta_{\omega_k}(z). \tag{2}$$

The set of those p for which the condition of the theorem is not fulfilled is at most countable.

Remark. It is easy to show that in the sum (2) only a finite number of terms are different from zero.

4. Let us apply this result to the equation

$$u(x) + \frac{1}{\pi} \int_\Gamma \frac{(\nu(y), x - y)}{|x - y|^2} \, u(y) \, d_y s = f(x), \tag{3}$$

corresponding to the interior Dirichlet problem for the Laplace equation. In this case

$$H^{(1)}_{\omega_k}(z) = H^{(2)}_{\omega_k}(z) = -\frac{\sin \omega_k}{\pi} \int_{-\infty}^{\infty} \frac{e^{izt} \, dt}{e^t + e^{-t} - 2\cos \omega_k} = -\frac{\operatorname{sh}(\pi - \omega_k) z}{\operatorname{sh} \pi z}.$$

When z runs along the line $\operatorname{Im} z = 1/p$, $1 < p < \infty$, the quantity $\operatorname{sh} |\pi - \omega_k| z / \operatorname{sh} \pi z$ describes, in the negative direction, a closed curve issuing from the origin and intersecting the right half-axis at the single point with abscissa

$$\frac{\sin |\pi - \omega_k| / p}{\sin \pi / p}.$$

This makes it possible to compute the index of equation (3).

Further, as follows from the results of I. Radon ([4]), equation (3), for any continuous right-hand side, has a continuous solution. Therefore, in those spaces L_p in which equation (3) is normally solvable, it is unconditionally solvable. Hence

Theorem 3. If $1 < p < \infty$ satisfies the conditions

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$$p \neq 1 + \frac{|\pi - \omega_k|}{\pi}, \qquad k = 1, 2, \dots, \tag{4}$$

then equation (3) is solvable in the space L_p for any right-hand side. The number of linearly independent solutions of the corresponding homogeneous equation is computed by the formula

$$\alpha_p = \frac{1}{2} \sum_{k=1}^{\infty} \left\{ 1 - \operatorname{sgn} \left(p - \frac{\pi + |\pi - \omega_k|}{\pi} \right) \right\}.$$

Corollary. If the curve Γ has a finite number of corner points, then for

$$1 < p < 1 + \min_k |\pi - \omega_k| / \pi$$

the number α_p is equal to the number of these points. If, however, the number of corner points of the curve Γ is infinite, then $\alpha_p \to \infty$ as $p \to 1$.

Introducing before the integral in (3) a complex multiplier λ, we see that, if conditions (4) are not fulfilled, then the operator in (3) is unstable with respect to small perturbations and, consequently, is not normally solvable. We also find that the following is true.

Theorem 4. The Fredholm radius (7) (see also (8)) of the operator

$$\frac{1}{\pi} \int_\Gamma \frac{(\nu(y), x - y)}{|x - y|^2} \, u(y) \, dy^s$$

in the space L_p, $1 < p < \infty$, is equal to

$$\Omega(p) = \min_k \frac{\sin \pi / p}{\sin |\pi - \omega_k| / p}.$$

Let us note that $\Omega(p) \to \min_k \pi / |\pi - \omega_k|$ as $p \to \infty$, which corresponds to the result of I. Radon ([4]).

5. Consider the equation obtained by Fredholm ([9]) for the second fundamental problem of the theory of elasticity, coinciding in the plane case with the Sherman-Lauricella equations ([10]):

$$u(x) + \frac{1}{\pi} \int_\Gamma \frac{(\nu(y), x - y)}{|x - y|^2} \left\{ (1 - \varkappa) E + 2\varkappa \left(\frac{x - y}{|x - y|} \right) \left(\frac{x - y}{|x - y|} \right)' \right\} u(y) \, dy^s = f(x). \tag{5}$$

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Here E is the identity matrix, the prime denotes passage from a column to a row, $\varkappa = (\lambda+\mu)/(\lambda+3\mu)$ (λ, μ are Lamé constants).

In this case ([3])

$$\Delta_{\omega_k}(z) = (1 - |W^+_{\omega_k}(z)|^2)(1 - |W^-_{\omega_k}(z)|^2),$$

where

$$W^{\pm}_{\omega_k}(z) = \frac{\operatorname{sh}(\pi-\omega_k)z}{\operatorname{sh}\pi z}(1 + z^2\varkappa^2\sin^2\omega_k)^{1/2} \pm \frac{\operatorname{ch}(\pi-\omega_k)z}{\operatorname{sh}\pi z} z\varkappa\sin\omega_k.$$

Having studied the curves described by the quantities $W^+_{\omega_k}(z)$, $W^-_{\omega_k}(z)$, when z runs over the straight lines $\operatorname{Im} z = 0$, $\operatorname{Im} z = 1/2$, we are convinced that the following holds.

Theorem 5. For equation (6) in the space L_p, $p \geqslant 2$, the Fredholm alternative is valid.

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Donetsk Computing Center
Academy of Sciences of the Ukrainian SSR

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