DIRECT SUMS OF DIVISION ALGEBRAS

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Abstract

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The classical theorem of Weierstrass–Dedekind is well known: Any nonzero finite-dimensional associative-commutative algebra without nilpotent elements over the field K of complex numbers is a direct sum of fields isomorphic to K.

In the present note it is proved that in this theorem one may replace the requirement of associativity-commutativity by the considerably weaker condition of associativity at zero, i.e.

$$x(yz) = 0 \Longleftrightarrow (xy)z = 0$$

for arbitrary elements x, y, z of the algebra under consideration. Moreover, instead of the field of complex numbers one may take any other algebraically closed field Φ.

Everywhere below, by an algebra we shall always mean a not necessarily associative finite-dimensional algebra $R \neq 0$ over an arbitrary algebraically closed field Φ.

Lemma 1. *Let A and B be $n \times n$ matrices over the field Φ, and suppose that the matrix A is nonsingular. There exists an element $\lambda \in \Phi$ such that the matrix $\lambda A + B$ is singular.*

Indeed, consider the polynomial*

$$f(x) = |xA + B| = \begin{vmatrix} a_{11}x + b_{11} & \dots & a_{1n}x + b_{1n} \\ \cdot & \cdot & \cdot \\ a_{n1}x + b_{1n} & \dots & a_{nn}x + b_{nn} \end{vmatrix}.$$

It is clear that

$$f(x) = |A|x^n + \dots + |B|.$$

But the field Φ is algebraically closed. Therefore, from $|A| \neq 0$ it follows that there exists $\lambda \in \Phi$ such that $f(\lambda) = 0$. This element $\lambda \in \Phi$ will be the required one.

Lemma 2. *Let the algebra R have no zero divisors. Then for any nonzero $a \in R$ the right multiplication $\rho(a)$, $\forall x \in R$, $x\rho(a) = xa$, is a nonsingular linear transformation of the linear space R over Φ.*

Let $\mathfrak{B}_R = \{e_i \mid 1 \le i \le n\}$ be a basis of the space R. Since for arbitrary $\alpha_i \in \Phi$ the equality

$$(\Sigma\alpha_i e_i)\rho(a) = \Sigma\alpha_i e_i a = (\Sigma\alpha_i e_i)a$$

holds, and R is an algebra without zero divisors, it follows that $\{e_i a \mid 1 \le i \le n\}$ is also a basis of the space R. Hence the linear transformation $\rho(a)$ is invertible, and therefore nonsingular.

Lemma 3. *The algebra R is a division algebra if and only if* $\dim R = 1$ *and* $R \approx \Phi$.

Recall that an algebra $R \neq 0$ is called a division algebra if and only if, for any nonzero $a, b \in R$, the equations $xa = b$ and $ay = b$ are solvable in R, and uniquely so. It is well known (see ([1]), p. 277) that division algebras are precisely algebras without zero divisors (in the finite-dimensional case!).

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* By $|A|$ is denoted the determinant of the matrix A. The matrix A is nonsingular if $|A| \neq 0$.

Let R be a division algebra, $\mathfrak{B}_R = \{e_i \mid 1 \le i \le n\}$ a basis of the space R. Suppose that $\dim R = n \ge 2$. Consider the right multiplications $\rho(e_i)$ and the corresponding matrices A_i.

By Lemma 2 all matrices A_i are nonsingular, since they correspond to the nonsingular linear transformations $\rho(e_i)$. Moreover, if among the elements $\alpha_i \in \Phi$ there are some different from zero, then the matrix $\sum \alpha_i A_i$ is nonsingular as well, since it corresponds to right multiplication by the element

$$a = \sum_{i=1}^{n} \alpha_i e_i \neq 0.$$

Using Lemma 1, we find an $\alpha_1 = \lambda \in \Phi$ such that the matrix $\alpha_1 A_1 + A_2$ is singular. Taking $\alpha_2 = 1$ and $\alpha_i = 0$ for all remaining i, we obtain, by what was said above, that the matrix $\sum_{i=1}^{n} \alpha_i A_i = \alpha_1 A_1 + A_2$ is nonsingular—a contradiction.

Consequently, $\dim R = n = 1$, i.e. $R = \Phi e_1$. But R is an algebra without zero divisors. Therefore $e_1^2 = \alpha e_1$ for some nonzero $\alpha \in \Phi$. Denoting $f = \alpha^{-1} e_1$, we obtain $f^2 = f$, and therefore the map $\gamma f \to \gamma$ is an isomorphism of the algebra R onto the field Φ. The lemma is proved.

Lemma 4. Let the following identities hold in the algebra R:

$$x^2 = 0 \Longleftrightarrow x = 0, \tag{*}$$

$$x(yz) = 0 \Longleftrightarrow (xy)z = 0. \tag{**}$$

Then, for any element $a \in R$, the sets

$$\{x \in R \mid xa = 0\}, \qquad \{x \in R \mid ax = 0\},$$

$$(0 : a)_R = \{x \in R \mid xa = ax = 0\}$$

coincide. Moreover, the annihilator $(0 : a)_R$ of the element a in the algebra R is an ideal.

Indeed, for any $x, y \in R$, $\alpha, \beta \in \Phi$, we obtain

$$xa = 0 \Rightarrow a(xa) = 0 \Rightarrow (ax)a = 0 \Rightarrow$$

$$\Rightarrow ((ax)a)x = 0 \Rightarrow (ax)^2 = 0 \Rightarrow ax = 0 \Rightarrow xa = 0,$$

$$xa = 0 \Rightarrow y(xa) = 0 \Rightarrow (yx) = 0,$$

$$xa = 0 \Rightarrow ax = 0 \Rightarrow (ax)y = 0 \Rightarrow$$

$$\Rightarrow a(xy) = 0 \Rightarrow (xy)a = 0,$$

$$\begin{matrix} xa = 0 \\ ya = 0 \end{matrix} \quad \Rightarrow \quad (\alpha x + \beta y)a = \alpha xa + \beta ya = 0.$$

Therefore the indicated sets coincide, and the annihilator $(0 : a)_R$ is an ideal of the algebra R.

Lemma 5. Let the identities $(*)$ and $(**)$ hold in the algebra R. If M is a minimal ideal of the algebra R, then the algebra M has no zero divisors.

Denote by $(x)_R$ the principal ideal of the algebra R generated by the element $x \in R$. Let M be a minimal ideal of the algebra R. Suppose that M is not an algebra without zero divisors. Then there exist nonzero elements $a, b \in M$ such that $ab = 0$.

It is clear that $(0:a)_R \cap M \neq 0$ and $(0:b)_R \cap M \neq 0$. By the minimality of M, taking Lemma 4 into account, we obtain

$$(0:a)_R \cap M = (0:b)_R \cap M = M(a)_R = (b)_R.$$

But then $a^2 = b^2 = 0$, and therefore $a = b = 0$ by $(*)$. We have obtained a contradiction. Consequently, the algebra M has no zero divisors. The lemma is proved.

Theorem. Let R be a nonzero finite-dimensional algebra over an algebraically closed field Φ. If in the algebra R the identities ···

properties

$$x^2 = 0 \Longleftrightarrow x = 0, \qquad x(yz) = 0 \Longleftrightarrow (xy)z = 0,$$

then R decomposes into a direct sum of fields isomorphic to Φ,

$$R = \sum_{i=1}^{n} \oplus R_i, \qquad R_i \approx \Phi, \qquad n = \dim R.$$

In particular, the algebra R is associative-commutative.

Proof. Choose in R some minimal ideal M_1. By Lemmas 3, 5, the algebra M_1 is isomorphic to the field Φ, and $\dim M_1 = 1$. In particular, M_1 is a field and $M_1 = \Phi e_1$, where e_1 is the identity element of the field M_1. We note that for any $x \in R$,

$$M_1 = Re_1 = \Phi e_1,$$

$$(x - xe_1)e_1 = xe_1 - (xe_1)e_1 = xe_1 - xe_1 = 0,$$

since M_1 is an ideal in R and e_1 is the identity of the field M_1. This means that $x - xe_1 \in (0:e_1)_R$. But $x = (x - xe_1) + xe_1$, and therefore the equality

$$R = M_1 \oplus (0:e_1)_R$$

holds, since, obviously, $\Phi e_1 \cap (0:e_1)_R = 0$.

If $(0:e_1)_R = 0$, then $R = M_1 \approx \Phi$, and there is nothing to prove. If, however, $(0:e_1)_R \neq 0$, then choose some minimal ideal $M_2 \subseteq (0:e_1)_R$. As above, M_2 is a field and $M_2 = \Phi e_2$, where e_2 is the identity of the field M_2. But then, repeating almost word for word for M_2 the arguments already given for M_1, we obtain

$$R = M_1 \oplus M_2 \oplus (0 : (e_1 + e_2))_R, \qquad M_1 \approx \Phi \approx M_2.$$

Continuing the process, at the n-th step, where $n = \dim R$, we obtain

$$R = M_1 \oplus M_2 \oplus ... \oplus M_n, \qquad M_i \approx \Phi.$$

This completes the proof of the theorem.

It is clear that the theorem of Weierstrass–Dedekind follows from the theorem just proved, since the field K of complex numbers is algebraically closed.

Let us give an example showing that condition $(**)$ is essential. Let R be a two-dimensional algebra over Φ with basis $\mathfrak{B}_R = \{e_1, e_2\}$ and defining relations

$$e_1^2 = e_1, \qquad e_2^2 = e_1, \qquad e_1 e_2 = 0, \qquad e_2 e_1 = e_2.$$

It is easy to see that R is a simple algebra. Moreover, if $a = \alpha e_1 + \beta e_2$, then $a^2 = (\alpha^2 + \beta^2) e_1 + \beta\alpha e_2$, and therefore

$$a^2 = 0 \Rightarrow \alpha^2 + \beta^2 = \beta\alpha = 0 \Rightarrow \alpha = \beta = 0 \Rightarrow a = 0,$$

i.e., in R the conditional identity $(*)$ holds. The conditional identity $(**)$ does not hold, since

$$(e_2 e_2) e_2 = e_1 e_2 = 0 \neq e_2 = e_2 e_1 = e_2 (e_2 e_2).$$

The example of an algebra with zero multiplication shows that condition $(*)$ is also essential.

Finally, the example of the field of complex numbers, considered as an algebra over the field of real numbers, shows that algebraic closedness of the field Φ is necessary for the validity of Lemma 3 and of the theorem.

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1. A. G. **Kurosh**, *Lectures on General Algebra*, Moscow, 1962.

Note: Figure translations are in progress. See original paper for figures.