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Abstract

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In recent years, in connection with the intensive study of the dynamics of the atmosphere and ocean, interest in the theory of oceanic circulations has increased substantially; its foundations have been laid in a number of investigations. At present the greatest interest is in studies of baroclinic models of the ocean, which more fully reflect the principal features of the dynamics of marine currents. The most substantive mathematical models of currents in a baroclinic ocean are considered in works ($^{1-3}$). In the present paper a method is proposed for the numerical solution of problems of oceanic circulations, based on splitting the complicated operators of the problem into simpler ones.

Consider an ocean in the form of a cylinder of constant depth H. Let, in the (x, y) plane, the coastline be a polygon with sides parallel to the axes x and y; the y-axis is directed northward, x eastward, and z vertically downward. Then, in the linearized formulation, we arrive at the following problem (^{1}):

$$\frac{\partial u}{\partial t} - lv = -\frac{1}{\bar\rho}\frac{\partial p}{\partial x}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$
$$\frac{\partial v}{\partial t} + lu = -\frac{1}{\bar\rho}\frac{\partial p}{\partial y}, \qquad \frac{\partial \rho}{\partial t} + \Gamma w = 0, \qquad \frac{\partial p}{\partial z} = g\rho, \tag{1}$$

where p and ρ are the deviations of pressure and density from the standard values $\bar p(z)$, $\bar\rho(z)$, and $\Gamma = d\bar\rho/dz$ is a prescribed function of depth. The remaining notation used in (1) is standard. Following works ($^{1-5}$), we adjoin to the system of equations (1) the boundary conditions

$$\partial p/\partial t + g\bar\rho w = 0 \quad \text{at } z = 0; \qquad w = 0 \quad \text{at } z = H, \tag{2}$$
$$\mathbf{un} = 0 \quad \text{on } S, \tag{3}$$

where S is the cylindrical surface whose generator is the coastline, and $\mathbf{n}$ is the outward normal to S. As initial data at $t = 0$ we take

$$u = u^0, \qquad v = v^0, \qquad p = p^0. \tag{4}$$

Under the assumptions made above, the solution of problem (1)–(4) will be sought by means of the method of orthogonal expansions (^{1}). To this end we represent the components of the solution of the system in the form of Fourier series

$$\begin{pmatrix} p \\ u \\ v \end{pmatrix} = \sum_{m=0}^{\infty} \begin{pmatrix} p_m \\ u_m \\ v_m \end{pmatrix} \psi_m(z), \qquad w = \sum_{m=0}^{\infty} w_m \frac{1}{\Gamma}\frac{d\psi_m}{dz}, \qquad \rho = \sum_{m=0}^{\infty} \rho_m \frac{d\psi_m}{dz}, \tag{5}$$

where $\psi_m(z)$ is the complete set of nontrivial solutions of the following spectral problem:

$$\frac{d}{dz}\frac{1}{\Gamma}\frac{d\psi}{dz} + \lambda\psi = 0, \tag{6}$$

$$\frac{1}{\Gamma}\frac{d\psi}{dz} - \frac{1}{\rho}\psi = 0 \quad \text{at } z = 0; \qquad \frac{1}{\Gamma}\frac{d\psi}{dz} = 0 \quad \text{at } z = H.$$

In [1] it was shown that problem (6) determines a basis of functions $\psi_m(z)$ and a system of positive eigenvalues λ_m. In [5] graphs of the functions $\psi_m(z)$ are given for typical cases of baroclinic stratifications. Substituting (5) into (1)–(4), we arrive at a set of problems for the Fourier coefficients—the functions u_m, v_m, w_m, p_m, and ρ_m, which no longer depend on z:

$$\frac{\partial u_m}{\partial t} - l v_m = -\frac{1}{\rho}\frac{\partial p_m}{\partial x}, \qquad \frac{\partial v_m}{\partial t} + l u_m = -\frac{1}{\rho}\frac{\partial p_m}{\partial y}, \tag{7}$$

$$\lambda_m \frac{\partial p_m}{\partial t} + \frac{\partial u_m}{\partial x} + \frac{\partial v_m}{\partial y} = 0 \quad (m = 1, 2, \ldots),$$

with boundary conditions

$$\mathbf{u}_m \mathbf{n} = 0 \quad \text{on } \sigma, \tag{8}$$

where σ is the shoreline contour in the xy-plane bounding the basin. The initial conditions will be

$$u_m = u_m^0, \qquad v_m = v_m^0, \qquad p_m = p_m^0. \tag{9}$$

We proceed to consider a numerical algorithm for solving problem (7)–(9). For this purpose we use the following approximation on the interval $0 \leq t \leq \tau$ and, omitting the index m, obtain

$$\frac{u-u^0}{\tau} - lv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \qquad \frac{v-v^0}{\tau} + lu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad \lambda\frac{p-p^0}{\tau} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{10}$$

Here the functions u, v, and p without superscripts refer to the time $t = \tau$. One could have used a second-order approximation in τ, but this is no longer connected with any fundamental difficulties. Eliminating the unknowns u and v from system (10) and assuming l to depend only on y, by the methods developed in [6], we arrive at the equation

$$\Delta p + bp_x - \mu^2 p = -f, \tag{11}$$

where

$$f = \mu^2 p^0 - \frac{\rho}{\tau}(D^0 + l\tau\Omega^0), \qquad b = \tau\frac{dl}{dy}, \qquad \mu^2 = \lambda l^2 \rho \frac{1+l^2\tau^2}{l^2\tau^2}, \qquad D^0 = \frac{\partial u^0}{\partial x} + \frac{\partial v^0}{\partial y},$$

$$\Omega^0 = \frac{\partial v^0}{\partial x} - \frac{\partial u^0}{\partial y}.$$

In deriving equation (11), the fact was used that the effect of variation of the Coriolis force with latitude is significant only for the evolution of large-scale and slowly occurring processes [6], i.e., for $l\tau \gg 1$. With the aid of system (10), the boundary condition can be written in a form resolved with respect to p. Indeed, on all parts of the boundary σ parallel to the y-axis we have

$$p_x + l\tau p_y = p_x^0 \quad \text{on } \sigma_1, \tag{12}$$

and on parts parallel to the x-axis,

$$p_y - l\tau p_x = p_y^0 \quad \text{on } \sigma_2. \tag{13}$$

In order that the boundary conditions (12) and (13) be compatible, we assume that at all corner points the simultaneous fulfillment of

$$p_x = 0, \qquad p_y = 0. \tag{14}$$

Relations (14) express the fact that in the neighborhood of the corner points $u = v = 0$. Problem (11)–(14) is the classical Poincaré problem, to which, under considerably more general assumptions, a large series of investigations has been devoted ([7–9] and others). It is not difficult to verify that the problem under consideration has defect index equal to zero and, consequently, belongs to the class of Fredholm-type problems [8, 9]. Trans-

we proceed to the formulation of a numerical algorithm. For this purpose, we cover the domain of definition of the solution of problem (11)–(14) by a uniform square grid with step $\Delta x = \Delta y = h$, assuming that the boundaries σ_1 and σ_2 coincide with segments of the coordinate lines of this grid. Next we extend our solution beyond the domain of definition of the solution by one step, assuming sufficient smoothness of the solution of the problem. Since the pressure on the boundary σ is unknown, we shall determine it as a result of solving the problem. First we exclude from consideration the points extrapolated beyond the domain of definition of the solution of the problem. To this end we proceed as follows. Fix some grid point on the boundary and write equation (11) in a difference form in a neighborhood of this point, using the second order of approximation. To this equation we adjoin a difference analogue of the boundary condition, of second order of accuracy, also written for the fixed point. Then, from the resulting system of two difference equations, we eliminate the unknown at the fictitious point. As a result we arrive at a difference equation that contains the unknowns at four neighboring points, including three boundary ones. Difference analogues for the equations in a neighborhood of corner points are constructed in an analogous way. In this case, to the difference equation are adjoined second-order accurate difference analogues of the two boundary conditions (14), and then the values of p at two fictitious points are eliminated. It is important to note that, under such a construction, the five-point structure of the difference analogue of the elliptic operator inside the domain of definition and on the boundary is completely preserved; only the coefficients of the unknowns change when boundary points of the domain σ are considered. As a result we arrive at a system of linear algebraic equations of the form

$$\Lambda\varphi = g, \tag{15}$$

where φ and g are vectors whose components are the values of p at all interior and boundary points of the grid domain, and Λ is the coefficient matrix. Of fundamental importance is the fact that the matrix Λ can be represented as the sum of two matrices $\Lambda = \Lambda_1 + \Lambda_2$, where the matrix Λ_1, acting on the vector φ, generates a set of independent systems of equations of Jacobi type, each of which combines only the solutions along each row, while Λ_2 generates systems of solutions only along columns; the inversion of such systems is efficient with the aid of factorization.

Thus, in our case the Poincaré problem has been reduced to a system of equations of the form

$$(\Lambda_1 + \Lambda_2)\varphi = g. \tag{16}$$

We now consider the grid Euclidean space of vectors $\varphi \in R$ with scalar product.

Lemma 1. The matrices Λ_1 and Λ_2 are positive, i.e. $(\Lambda_1\varphi, \varphi) > 0$, $(\Lambda_2\varphi, \varphi) > 0$.

We shall find the solution of equation (16) by means of the iterative method (4)

$$B_j \frac{\varphi^{j+1} - \varphi^j}{\tau_j} + \Lambda\varphi^j = g, \qquad \varphi^0 = 0, \tag{17}$$

where

$$B_j = \left(E + \frac{\sigma_j}{2}\Lambda_1\right)\left(E + \frac{\sigma_j}{2}\Lambda_2\right),$$

and τ_j and σ_j are for the time being arbitrary positive parameters of the relaxation process. We choose the parameter τ_j, for fixed σ_j, on the basis of the method of minimal residuals (^{10}), which we modify as follows. Introduce the residual vector $\xi^j = \Lambda\varphi^j - g$. Then for the residual ξ^j we obtain the equation

$$\xi^{j+1} = \xi^j - \tau_j \Lambda B_j^{-1}\xi^j, \qquad \xi^0 = -g. \tag{18}$$

Let us now consider the scalar product (ξ^{j+1}, ξ^{j+1}) and, taking (18) into account, we shall have

$$(\xi^{j+1}, \xi^{j+1}) = q_j(\xi^j, \xi^j), \tag{19}$$

where

$$q_j = 1 - 2\tau_j \frac{(\Lambda B_j^{-1}\xi^j, \xi^j)}{(\xi^j, \xi^j)} + \tau_j^2 \frac{(\Lambda B_j^{-1}\xi^j, \Lambda B_j^{-1}\xi^j)}{(\xi^j, \xi^j)}.$$

From the minimum condition for $q_j(\tau)$ we obtain

$$\tau_j = \frac{(\Lambda B_j^{-1}\xi^j, \xi^j)}{(\Lambda B_j^{-1}\xi^j, \Lambda B_j^{-1}\xi^j)}. \tag{20}$$

Let us consider the scheme for realizing the iterative process. Denote $y^{j+1} = B_j^{-1}\xi^j$, $z^{j+1} = \Lambda y^{j+1}$. Then, for the given ξ^j, the auxiliary function y^{j+1} is found from the equation

$$B_j y^{j+1} = \xi^j. \tag{21}$$

This equation reduces to the system

$$\left(E+\frac{\sigma_j}{2}\Lambda_1\right)y^{j+1/2}=\xi^j, \qquad \left(E+\frac{\sigma_j}{2}\Lambda_2\right)y^{j+1}=y^{j+1/2}, \tag{22}$$

After this the auxiliary function z^{j+1} and the parameter are found:

$$\tau_j=\frac{(z^{j+1},\xi^j)}{(z^{j+1},z^{j+1})}. \tag{23}$$

The new approximation for φ^{j+1} is found in the form

$$\varphi^{j+1}=\varphi^j-\tau_j y^{j+1}. \tag{24}$$

Up to this point it has been assumed that the parameter σ_j is arbitrary. It is expedient to choose this parameter in the form $\sigma_j=\tau_{j-1}$. As the initial operator B_0 we choose a form independent of σ_1: $B_0=\Lambda_1\Lambda_2$.

Theorem. *Taking Lemma 1 into account, the iterative process (17) converges to the exact solution of system (15).*

The proof of the theorem can be carried out following the works ([10,11]).

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Note: Figure translations are in progress. See original paper for figures.