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Abstract

Full Text

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ON THE ARRANGEMENT OF THE OVALS OF A CURVE OF THE SIXTH ORDER

(Presented by Academician I. G. Petrovskii, 5 VII 1968)

MATHEMATICS

1. A nonsingular plane real curve of the 6th order C_6^* can have, by Harnack ⁽¹⁾, at most 11 ovals. We shall say that a curve C_6 has type $\frac{k}{1}l$ (or is a C_6 -curve $\frac{k}{1}l$) if it has a certain oval α (the principal one), inside which there lie k ovals exterior to one another and l ovals outside the oval α and exterior to one another (see ⁽¹¹⁾). The type of a curve C_6 consisting of k ovals exterior to one another will be denoted by k . A curve C_6 can have the following logically possible types with 11, 10, and 9 ovals:

Table 1

$$\begin{array}{cccccccccccc} \frac{10}{1}; & \frac{9}{1}1; & \frac{8}{1}2; & \frac{7}{1}3; & \frac{6}{1}4; & \frac{5}{1}5; & \frac{4}{1}6; & \frac{3}{1}7; & \frac{2}{1}8; & \frac{1}{1}9; & 11, \\ \frac{9}{1}; & \frac{8}{1}1; & \frac{7}{1}2; & \frac{6}{1}3; & \frac{5}{1}4; & \frac{4}{1}5; & \frac{3}{1}6; & \frac{2}{1}7; & \frac{1}{1}8; & 10, \\ \frac{8}{1}, & \frac{7}{1}1; & \frac{6}{1}2; & \frac{5}{1}3; & \frac{4}{1}4; & \frac{3}{1}5; & \frac{2}{1}6; & \frac{1}{1}7; & 9 \end{array}$$

A. Harnack constructed a curve C_6 of type $\frac{1}{1}9$ ⁽¹⁾, D. Hilbert one of type $\frac{9}{1}1$ ⁽²⁾. D. Hilbert, in formulating his 16th problem ⁽³⁾, expressed the conviction that from the first row of Table 1 there exist only curves of the types $\frac{9}{1}1$ and $\frac{1}{1}9$. H. Kahn ⁽⁴⁾ and K. Lebenstein ⁽⁵⁾ tried to prove (without success) that a curve C_6 of type 11 does not exist. K. Rohn tried to prove the nonexistence of curves C_6 of types $\frac{10}{1}$ and 11 ⁽⁶⁾, but there are errors in his work. The nonexistence of a curve C_6 of type 11 follows from the theorem of I. G. Petrovskii ⁽⁹⁾. We asserted ⁽¹¹⁾ that a curve cannot have the types situated above the broken line in Table 1, and also the types $\frac{5}{1}3$, $\frac{5}{1}4$, and $\frac{5}{1}5$. However, in the proofs (which were not published) of the nonexistence of curves C_6 of types $\frac{5}{1}3$, $\frac{5}{1}4$, $\frac{5}{1}5$, $\frac{6}{1}3$, $\frac{6}{1}4$, we

made errors. After correcting the errors and carrying out additional analysis it turned out that curves $\frac{6}{1}3$ and $\frac{6}{1}4$ do not exist, the curve $\frac{5}{1}3$ is constructed by D. Hilbert's method, and the curves $\frac{5}{1}4$ and $\frac{5}{1}5$ exist, but are not constructed by the methods of A. Harnack and D. Hilbert.

Thus, curves C_6 of the types situated below the broken line in Table 1 exist, while those above the broken line do not exist. (For complete proofs of this assertion see ⁽¹³⁾.)

* By C_m, \overline{C}_m , etc., in this note is denoted a plane real curve of the m -th order.

2. In the present note we shall set forth the idea of constructing a curve of type $\frac{5}{1}5$ (and consequently also $\frac{5}{1}4$).

Lemma 1. Let $F(x)$ be an irreducible real simple (i.e., all its singular points are simple double points) curve of degree 6, let y_0, y_1, y_2 be singular points of the curve F , and let the lines $x_i \equiv y_k y_l$ ($i, k, l = 0, 1, 2$ and distinct) not be tangent to the curve F . Then, under the quadratic transformation

$$x_0 = y_1 y_2, \quad x_1 = y_0 y_2, \quad x_2 = y_0 y_1 \quad (1)$$

(with fundamental points y_0, y_1, y_2 and fundamental lines x_0, x_1, x_2), the curve $F(x)$ is transformed in the plane (y) into a curve $F_{(1)}(y)$ —an irreducible real simple curve of degree 6, having at the points x_0, x_1, x_2 simple double points. The remaining singular points of the curve $F(x)$ (except for y_0, y_1, y_2) are transformed into the same kind of singular points of the curve $F_{(1)}(y)$, and the transformation (1) establishes a one-to-one and continuous correspondence between the branches of the curves F and $F_{(1)}$.

The proof follows easily from the theorems of § 7, Chapter 3 ⁽¹⁰⁾.

Lemma 2. Let F be a simple real unicursal curve of degree 6 and let all its ⁽¹⁰⁾ singular points be real.

Then, by an arbitrarily small change of the coefficients of the curve F , one can obtain a curve Φ of degree 6 isotopic to F , for which the line through any two singular points is not tangent to the curve Φ .

The proof is not difficult.

Theorem 1. There exists a curve C_6 of type $\frac{5}{1}5$.

Proof. Let C_3 be a unicursal curve with a node z_1 (see Fig. 1a). On C_3 there exists a point z_2 such that on the arc $z_1 z_2$ there are no inflection points of the curve C_3 . By rotating the line $z_1 z_2$ about the point z_2 , we obtain a line L_1 , intersecting C_3 further at the points z_3 and z_4 on the loop of the curve

Fig. 1

Figure 1: Fig. 1

C_3 . The tangent to C_3 at the point z_2 intersects C_3 further at the point Q . We may assume that z_2 is so close to z_1 that the whole portion of the curve C_3 Qz_1z_2 lies in the finite part of the plane. Next, by rotating the line z_1z_2 about the point O (lying on the exterior segment z_1z_2), we obtain a line L_2 , intersecting the tangent Qz_2 at a point P , lying on the finite segment Qz_2 , and C_3 at the points z_6, z_7 (on the arc z_1z_2) and z_5 . L_2 intersects L_1 at the point z_8 . Finally, let us move the tangent z_2Q into the position L_3 in the following way: move the point of tangency to z_9 (in the direction from z_7 to z_2), then rotate the tangent to C_3 at z_9 about z_9 so as to obtain a line intersecting C_3 at a point z_{10} , close to z_9 and situated on the other side of z_9 than the point z_2 , and also at some point z_{11} (close to Q). L_3 intersects the lines L_1 and L_2 at the points z_{12} and z_{13} . Observe that, under the displacement of Qz_2 into L_3 , the point P passes into z_{13} . This displacement may be assumed so small that the line z_2z_{13} has with C_3 , on the arc z_1Oz_{11} , two points of intersection. The curve $C_3L_1L_2L_3 \equiv \Phi$ is a simple reducible curve of degree 6 with 13 nodes. By Theorem 10, § 6 ⁽¹²⁾, one can bifurcate the nodes z_8, z_4 , and z_{10} and preserve the remaining nodes of the curve Φ , so as to obtain a simple unicursal curve F of Fig. 1b. Choose the fundamental points y_0, y_1, y_2 of the transformation (1) at z_2, z_{13} , and z_{12} . In the plane (y) we obtain (by Lemmas 1 and 2) a curve $F_{(1)}(y)$ with new nodes x_0, x_1, x_2 . The nodes z_2, z_{13} , and z_{12} disappear, while the remaining nodes of the curve $F(x)$ are preserved, and we denote them by the former letters. Then subject the curve $F_{(1)}(y)$ to the transformation (1), choosing as fundamental points y'_0, y'_1, y'_2 the points z_5, z_1 , and z_3 . Let x'_0, x'_1, x'_2 be the corresponding fundamental lines. Then in the plane (y') we obtain a curve $F_{(2)}(y')$ with simple double points $x'_0, x'_1, x'_2; x_0, x_1, x_2; z_6, z_9, z_{11}$. Choose the fundamental points y''_0, y''_1, y''_2 of the transformation (1) of the curve $F_{(2)}(y')$ at the points x_0, z_9, z_{11} , and denote the fundamental lines by x''_0, x''_1, x''_2 . In the plane (y'') , we obtain a curve $F_{(3)}(y'')$ with singular points $x''_0, x''_1, x''_2; x'_0$,

$x'_1, x'_2; x_1, x_2, z_6, z_7$. Finally, for the curve $F_{(3)}(y'')$ we choose the fundamental points y'''_0, y'''_1, y'''_2 of the transformation (1) at the points x'_0, z_6, z_7 , and denote the fundamental lines through them by x'''_0, x'''_1, x'''_2 . In the plane (y''') we obtain the curve $F_{(4)}(y''')$ —a simple unicursal curve of the 6th order (Fig. 1) of type $\frac{5}{1}5$. Under the last three quadratic transformations the lines $x'_1, x'_2; x''_0, x''_1, x''_2$ and x'''_0, x'''_1, x'''_2 may intersect the corresponding curves $F_{(1)}, F_{(2)}$, and $F_{(3)}$ in two real points. Then the corresponding points of the curve in Fig. 1 will be nodes analogous to the nodes x_1, x_2 . The curve $F_{(4)}$ in this case always has type $\frac{5}{1}5$. The theorem is proved.

Fig. 1

Theorems 2 and 3 can be proved.

Theorem 2.

1°. A curve C_6 of type $\frac{9}{1} 1$ can be obtained from a simple reducible curve only of type $C_2 \cdot C_4$.

2°. A curve C_6 of type $\frac{5}{1} 5$ can be obtained from a simple reducible curve only of type $C_1 \cdot C_5$ (Fig. 1a).

3°. A curve C_6 of type $\frac{1}{1} 9$ can be obtained from simple reducible curves of each of the types $C_2 \cdot C_4$; $C_1 \cdot C_5$; $C_3 \cdot C_3$.

A. Wiman ⁽⁷⁾ showed that there exist curves of even orders $m \geq 8$ with the greatest number of ovals (M -curves) which are not constructed by the methods of Harnack and Hilbert (see also ⁽⁸⁾, Chapter 4, § 2).

Theorem 3. *From the curve $C_6 = C_1 \cdot C_5$ of Fig. 1, by applying Harnack's construction, one obtains a series of M -curves. In particular, the following curves are obtained:*

1) For odd $m \geq 7$, a curve of type

$$I \frac{5}{1} \left\{ \frac{m^2 - 3m}{2} - 5 \right\},$$

i.e., having an odd branch I, five ovals inside one of them, and $\frac{m^2 - 3m}{2} - 5$ exterior ovals (outside one another).

2) For even $m \geq 6$, a curve of type

$$\frac{5}{1} \frac{\left\{ \frac{1}{8} m(m - 6) \right\}}{1} \left\{ \frac{3m^2 - 6m}{8} - 4 \right\},$$

which are not constructed by the methods of A. Harnack, D. Hilbert, and A. Wiman ^(1,2,7).

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