ON A CERTAIN TRANSFER THEOREM

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 519.21

MATHEMATICS

Academician of the Academy of Sciences of the Ukrainian SSR B. V. GNEDENKO, HUSSEIN FAHIM

ON A CERTAIN TRANSFER THEOREM

A large number of mathematical and applied problems lead to the necessity of summing independent random variables in a random number and of studying limit distributions for such sums. After the papers ([1,2]) this topic attracted the attention of many researchers. Here we shall note only the works ([3–5]). In ([4,5]) there is a fairly complete bibliography.

Consider a sequence of integer-valued random variables $\{\nu_n\}$ and an array of random variables ξ_{nk} with two entries. We shall assume that for each n the variables ξ_{nk} are identically distributed and independent in the aggregate. Suppose further that, for each n, the variable ν_n is independent of the variables of the sequence $\{\xi_{nk}\}$.

Theorem. If there exists a sequence $\{k_n\}$ such that, as $n \to \infty$:

a)

$$\mathbf{P}\left\{\sum_{k=1}^{k_n} \xi_{nk} < x\right\} \to \Phi(x);$$

b) with a suitable choice of constants c_n

$$\mathbf{P}\left\{\frac{\nu_n - k_n}{c_n} < x\right\} \to A(x),$$

where $\Phi(x)$ and $A(x)$ are distribution functions;

c) $c_n/k_n \to r$ $(0 < r < \infty)$,

then the distributions of the sums

$$S_{\nu_n} = \xi_{n1} + \xi_{n2} + ... + \xi_{n\nu_n}$$

as $n \to \infty$ converge to a limit $\Psi(x)$. The characteristic function $\psi(t)$ of the limiting distribution is equal to

$$\psi(t) = \int_0^\infty [\varphi(t)]^z \, dA^*(z),$$

where $\varphi(t)$ is the characteristic function of the distribution $\Phi(x)$, and $A^*(z) = A(y - 1/r)$.

Proof. The characteristic function of the sum S_{ν_n} is equal to

$$\varphi_n(t) = \sum_{j=0}^{\infty} p_{nj} (f_n(t))^j,$$

where

$$p_{nj} = \mathbf{P}\{\nu_n = j\}, \qquad f_n(t) = \int_{-\infty}^{\infty} e^{itx} \, dF_n(x).$$

Put

$$A_n(x) = \mathbf{P}\{\nu_n < x\}.$$

It is obvious that

$$\varphi_n(t) = \int_0^\infty f_n^x(t) \, dA_n(x).$$

Let now

$$\overline{A}_n(y) = \mathbf{P}\left\{\frac{\nu_n - k_n}{c_n} < y\right\} = A_n(c_{ny} + k_n).$$

In this notation

$$\varphi_n(t) = \int_{-k_n/c_n}^{\infty} f^{yc_n + k_n}(t) \, d\overline{A}_n(y) = \int_{-k_n/c_n}^{\infty} \left[f^{k_n}(t)\right]^{yc_n/k_n + 1} \, d\overline{A}_n(y).$$

According to the assumptions of the theorem and the known assumptions on passing to the limit under the integral sign, in every finite interval of t

$$\psi(t) = \lim_{n \to \infty} \varphi_n(t) = \int_{-r^{-1}}^{\infty} \varphi^{ry+1}(t) \, dA(y).$$

If now we denote $z = ry + 1$, then we arrive at the assertion of the theorem.

Since, according to a well-known theorem of A. Ya. Khinchin ((6), p. 197), $\varphi(t)$ is the characteristic function of an infinitely divisible distribution and, with a suitable choice of $F(x)$ and $\{k_n\}$, one can obtain any infinitely divisible characteristic function, then

$$\varphi(t) = e^{v(t)},$$

where γ is an arbitrary real constant, $G(x)$ is a nondecreasing function of bounded variation, and

$$v(t) = i\gamma t + \int_{-\infty}^{\infty} \left\{ e^{itx} - 1 - \frac{itx}{1+x^2} \right\} \frac{1+x^2}{x^2}\, dG(x).$$

Since

$$\psi(t) = \int_0^\infty e^{iy(-iv(t))}\, dA^*(y) = a(-iv(t)),$$

where

$$a(t) = \int_0^\infty e^{itx}\, dA^*(x),$$

we conclude that if $a(t)$ is the characteristic function of a random variable taking only nonnegative values, and $v(t)$ is the logarithm of the characteristic function of an infinitely divisible distribution, then the function

$$\psi(t) = a(-iv(t))$$

is characteristic. Moreover, according to a remark of W. Feller ((7), p. 646), if the function $A^*(y)$ is infinitely divisible, then $\psi(t)$ is the characteristic function of an infinitely divisible distribution.

Example 1. If ν_n has a geometric distribution, then

$$A^*(z) = 1 - e^{-z} \qquad (z \geqslant 0).$$

The functions

$$\psi(t) = \int_0^\infty e^{-z(1-v(t))}\, dz = \frac{1}{1-v(t)},$$

according to what has been proved, are characteristic and infinitely divisible, since the distribution $1 - e^{-z}$ is infinitely divisible.

Example 2. Characteristic and at the same time infinitely divisible are all functions

$$\psi(t) = \frac{1}{1 + c|t|^\alpha \left\{1 + i\beta\frac{t}{|t|}\omega(t,\alpha)\right\} + i\gamma t},$$

where the constants $c > 0,\ -1 \le \beta \le 1,\ 0 < \alpha \le 2,\ -\infty < \gamma < \infty$, and

$$\omega(t,\alpha) = \begin{cases} \operatorname{tg}\dfrac{\pi}{2}\alpha, & \text{for } \alpha \ne 1, \\ \dfrac{2}{\pi}\ln|t|, & \text{for } \alpha = 1. \end{cases}$$

The proof that the functions

$$\psi(t) = \frac{1}{1 + |t|^\alpha}, \qquad 0 < \alpha \le 2,$$

are characteristic was given in the work of Yu. V. Linnik ([8]). Our example supplements this result in two directions.

Example 3. All functions

$$\psi(t) = \left(\frac{1}{1 - v(t)}\right)^\alpha,$$

where $\alpha > 0$, and $v(t)$ is the logarithm of the characteristic function of an infinitely divisible distribution, are characteristic and at the same time infinitely divisible.

For the proof it suffices to take

$$A^*(z) = \int_0^z \frac{x^{\alpha-1}}{\Gamma(\alpha)} e^{-x}\,dx \qquad (z \ge 0).$$

Example 4. If ν_n is uniformly distributed on the interval $(0; 2k_n)$, then as $n \to \infty$

$$\mathbf{P}\left\{\frac{\nu_n - k_n}{2k_n} < x\right\} \to \begin{cases} 0, & \text{for } x \le -0.5, \\ 0.5x, & \text{for } -0.5 \le x \le 0.5, \\ 1, & \text{for } x > 0.5. \end{cases}$$

Thus, all functions

$$\psi(t) = \int_0^1 e^{zv(t)}\,dz = \frac{e^{v(t)} - 1}{v(t)},$$

where $v(t)$ is the logarithm of the characteristic function of an infinitely divisible distribution, are characteristic (but not necessarily infinitely divisible).

In particular, the functions

$$\psi(t) = \frac{1 - e^{-|t|^\alpha}}{|t|^\alpha}$$

are characteristic for all constants $\alpha,\ 0 < \alpha \leq 2$.

Moscow State University
named after M. V. Lomonosov

Received
22 III 1969

CITED LITERATURE

[1] H. Robbins, *Proc. Nat. Acad. Sci. U.S.A.*, **34**, 162 (1948).
[2] H. Robbins, *Bull. Am. Math. Soc.*, **54**, 1151 (1948).
[3] R. L. Dobrushin, *UMN*, **10**, issue 2 (64), 157 (1955).
[4] W. Richter, *Math. Nachr.*, **29**, Heft 5-6, 347 (1965).
[5] S. Guiașu, *Studii și cercetări Matem.*, **19**, fasc. 7, 971 (1967).
[6] B. V. Gnedenko, A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Moscow-Leningrad, 1949.
[7] V. Feller, *An Introduction to Probability Theory and Its Applications*, 2, Moscow, 1967.
[8] Yu. V. Linnik, *Ukr. Mat. Zh.*, **5**, issues 2-3, 207, 247 (1953).

Note: Figure translations are in progress. See original paper for figures.