A PRIORI ESTIMATES AND HYPOELLIPTIC OPERATORS WITH MULTIPLE CHARACTERISTICS

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Abstract

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A PRIORI ESTIMATES AND HYPOELLIPTIC OPERATORS WITH MULTIPLE CHARACTERISTICS

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In the paper ([1]) L. Hörmander, for a second-order differential operator of the form

$$Pu = \sum_{j=1}^{N} X_j^2 u + iX_0 u + \gamma(x)u, \tag{1}$$

where $X_j(x, D)$ $(j = 0, 1, \dots, N)$ are first-order differential operators with infinitely differentiable coefficients, obtained sufficient conditions for the existence of estimates

$$\sum_{j=1}^{N} \|X_j u\|_{(\varepsilon)}^2 + \|u\|_{(2\varepsilon)}^2 \leq C(K)\{\|Pu\|_{(0)}^2 + \|u\|_{(0)}^2\}, \qquad u \in C_0^\infty(K), \tag{2}$$

where K is any compact set in the domain $\Omega \subset R^n$ and the constant $\varepsilon = \varepsilon(K) > 0$ (see ([2])); it can be shown that Hörmander' s conditions are also necessary in the class of operators of the form (1) for the existence of the estimate (2) with some $\varepsilon(K) > 0$ $(D_j = -i\partial/\partial x_j,\ j = 1, \dots, n;\ i^2 = -1)$. In the paper ([2]) a simple proof of Hörmander' s theorem was obtained. Using the same method, we shall consider the general case of second-order differential operators with nonnegative characteristic form, namely:

$$Lu \equiv \sum_{k,j=1}^{n} a^{kj}(x) D_{kD} ju + iX_0(x, D)u + \gamma(x)u, \tag{3}$$

where $a^{kj}(x)\xi_k\xi_j \geq 0$ for any point $(x, \xi) \in \Omega \times S^{n-1}$.

The question of global smoothness of generalized solutions of boundary-value problems for equation (3) was considered in the papers ([3–6]). We shall give sufficient conditions for local smoothness of generalized solutions of equation (3).

In the domain $\Omega \subset R^n$ consider the system of differential operators

$$\{L^{0(j)},\ j=1,\dots,n;\quad L^0_{(j)},\ j=1,\dots,n;\quad X_0+L^{0(j)}_{(j)}\},$$

where

$$L^{0(j)}(x,D) \equiv \sum_k a^{kj}(x)D_k \qquad (j=1,\dots,n);$$

$$L^0_{(s)}(x,D) \equiv \sum_{k,j} \frac{\partial a^{kj}(x)}{\partial x_s} D_{kD}j \quad (s=1,\dots,n); \qquad L^{0(j)}_{(j)} \equiv \sum_{k,j} \frac{\partial a^{kj}}{\partial x_j} D_k.$$

For any multi-index $I=(\alpha_1,\dots,\alpha_k)$ $(k \geq 1)$, where $\alpha_s = 0,1,\dots,2n$, for any $s=1,\dots,k$ put: $|I| = \sum_{s=1}^k \lambda_s$, where $\lambda_s = 2$ if $\alpha_s = 0$, and $\lambda_s = 1$ if $\alpha_s = 1,\dots,2n$, and let

$$A_I(x,D) \equiv \operatorname{ad} A_{\alpha_1} \cdots \operatorname{ad} A_{\alpha_{k-1}} A_{\alpha_k},$$

where $A_s(x,D) \equiv L^{0(s)}(x,D)$ $(s=1,\dots,n)$; $A_{s+n}(x,D) \equiv L^0_{(s)}(x,D)$ $(s=1,\dots,n)$; $A_0(x,D) \equiv X_0 + L^{0(j)}_{(j)}$ and $\operatorname{ad} AB = AB - BA$ for any pseudodifferential operators A, B (see, for example, ([7])).

Definition. The system of operators $(A_0,\dots,A_{2n})$ has at the point $x_0 \in \Omega$ rank $R_{(A_0,\dots,A_{2n})}(x_0) = k$, if

$$\sum_{|I| \leq k-1} \left|A^0_I(x_0,\xi)\right| = 0 \quad \text{for some } \xi \in S^{n-1}; \tag{4}$$

$$\sum_{|I| \leq k} \left|A^0_I(x_0,\xi)\right| \neq 0 \quad \text{for every } \xi \in S^{n-1}, \tag{5}$$

where A^0_I is the principal part of the symbol $\sigma(A_I)(x,\xi)$ of the operator A_I (see, for example, (7)), $S^{n-1} = \{\xi \in R^n : \sum \xi_j^2 = 1\}$.

Remark. It is easy to show that the definition of $R_{(A_0,\dots,A_{2n})}(x)$ is invariant under infinitely differentiable changes of independent variables; moreover, if $R_{(A_0,\dots,A_{2n})}(x) < \infty$ for every point $x \in \Omega$, then

$$R_L(K) = \sup_K R_{(A_0,\dots,A_{2n})}(x) = k(K) < \infty$$

for every compact set $K \subset \Omega$.

Theorem 1. If $R_L(x) < \infty$ for every point $x \in \Omega$, then for every compact $K \subset \Omega$ there exists a constant $C(K)$ such that the inequality

$$\sum_{j=1}^{n} \left(\left\| L^{0(j)} u \right\|^2_{(\varepsilon(K))} + \left\| L^0_{(j)} u \right\|^2_{(\varepsilon(K)-1)} \right) + \|u\|^2_{(\varepsilon(K)+2^{1-R_L(K)})} \leq$$

$$\leq C(K) \left(\|Pu\|^2_{(0)} + \|u\|^2_{(0)} \right), \quad u \in C_0^\infty(K), \tag{6}$$

holds, where

$$\varepsilon(K) = \min \left(1,\ 2^{1-R(K)} \left(2^{R(K)-1} - 1 \right)^{-1} \right).$$

Theorem 2. If $R_L(x) < \infty$ for every point $x \in \Omega$, then for every compact $K \subset \Omega$, every $s \in R^1$, there exists a constant $C(K, s)$ such that for every function $u \in D'(\Omega)$ such that $Lu \in H^{\mathrm{loc}}_{(s)}$, an estimate of the form

$$\|\varphi u\|^2_{(s+\varepsilon(K)+2^{1-R_L(K)})} \leq C(K, s) \left(\|\varphi_1 Pu\|^2_{(s)} + \|\varphi_1 u\|^2_{(\gamma)} \right), \tag{7}$$

holds, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$ and $\varphi_1 \equiv 1$ in a neighborhood of $\operatorname{supp} \varphi$, $\gamma = \text{const} < s + \varepsilon(K) + 2^{1-R_L(K)}$.

Theorem 3. If $R_L(x) < \infty$ for every point x of the manifold M, where L is a second-order differential operator with nonnegative characteristic form, defined on M, then L is a hypoelliptic operator.

The proof of Theorem 1, from which Theorems 2 and 3 can be obtained by known methods (see, for example, (10)), is based on the following auxiliary assertions.

Lemma 1 (energy estimate). For every compact $K \subset \Omega$ and every $s \geq 0$ there exists a constant $C(K, s)$ such that for every μ $(0 < \mu < 1)$ the inequality

$$\sum_{j=1}^{n} \left(\left\| L^{0(j)} u \right\|^2_{(s)} + \left\| L^0_{(j)} u \right\|^2_{(s-1)} \right) + \left\| \left(X_0 + L^{0(j)}_{(j)} \right) u \right\|^2_{(s-1/2)} \leq$$

$$\leq C(K, s) \left\{ \frac{1}{\mu} \|Lu\|^2_{(0)} + \mu \|u\|^2_{(2s)} + C_\mu \|u\|^2_{(0)} \right\}, \quad u \in C_0^\infty(K). \tag{8}$$

For every $s \in R^1$ in the domain Ω introduce the pseudodifferential operator $\mathcal{E}_{(s)}$ with symbol

$$\sigma(\mathcal{E}_{(s)})(x, \xi) = \varphi(x)(1 + |\xi|^2)^{s/2},$$

where the function $\varphi(x) \in C_0^\infty(\Omega)$ and $\varphi \equiv 1$ in a neighborhood of the compact set K.

Consider the system $(Q_0, \dots, Q_{2n})$ of first-order pseudodifferential operators, where $Q_j = L^{0(j)}$ $(j = 1, \dots, n)$; $Q_0 = X_0 + L^{0(j)}_{(j)}$, $Q_{n+j} = L^0_{(j)} \mathcal{E}_{(-1)}$ $(j = 1, \dots, n)$.

Lemma 2. For every compact $K \subset \Omega$, every integer $k \geq 1$, and every s $(0 \leq s < 1/2^{k-1})$ there exists a constant $C(K, k, s)$ such that for

for any μ $(0 < \mu < 1)$ the inequality

$$\sum_{|I|=k} \|Q_{Iu}\|^2_{(s-1+2^{1-k})} \leq C(K, k, s) \times$$

$$\times \left\{ \frac{1}{\mu} \|Lu\|^2_{(0)} + \mu \|u\|^2_{(2s)} + \mu \|u\|^2_{(2^{k-1}s)} + C_\mu \|u\|^2_{(0)} \right\}, \quad u \in C_0^\infty(K). \tag{9}$$

From the condition $R_L(K) < \infty$ it follows that there exists a neighborhood $O_x \subset \Omega$ of the point x and a finite system of operators $(Q_{I_1}, \dots, Q_{I_l})$, $(|I_j| \leq R$ for any $j \leq l)$, which is elliptic in the domain U_x. Therefore the following is valid.

Lemma 3. If $R_L(x) < \infty$, then there exists a neighborhood $U_x \subset \Omega$ of the point x such that an estimate of the form

$$\|u\|^2_{(s+2^{1-R_L(x)})} \leq C_1 \left\{ \sum_{|I| \leq R_L(x)} \|Q_{Iu}\|^2_{(s-1+2^{1-|I|})} + \|u\|^2_{(s)} \right\}, \quad u \in C_0^\infty(O_x). \tag{10}$$

From Lemmas 1-3 follows the proof of Theorem 1.

In papers ([8],[9]), for a scalar pseudodifferential operator P and any compact set $K \subset \Omega$, algebraic necessary and sufficient conditions were obtained for the existence of estimates of the form:

$$\|u\|^2_{(m-\delta)} \leq C(K) \{ \|Pu\|^2_{(0)} + \|u\|^2_{(0)} \}, \quad u \in C_0^\infty(K), \tag{11}$$

where m is the order of the operator P and $0 \leq \delta < 3/4$.

We shall consider the class of pseudodifferential operators P of order m, satisfying the following conditions:

1. $p^0(x, \xi) \geq 0$ for any point $(x, \xi) \in \Omega \times S^{n-1}$.
2. $p^1(x, \xi) = -\overline{p^1(x, \xi)}$ for any point $(x, \xi) \in N$, where

   $$N = \{(x, \xi) \in \Omega \times S^{n-1};\ p^0(x, \xi) = 0\},$$

   and $p^\nu(x, \xi)$ are the terms homogeneous of order $m - \nu$ in the asymptotic expansion in ξ of the symbol $\sigma(P)(x, \xi)$ of the operator P as $\xi \to \infty$ (see, for example, (7)).

Applying the method of localization of pseudodifferential operators proposed in paper ([8]), we obtain algebraic necessary and sufficient conditions for the existence, for any compact set $K \subset \Omega$, in the class of operators satisfying conditions 1, 2, of estimates of the form

$$\|\varphi u\|^2_{(m+s-1)} \leq C(K,s)\{\|\varphi_1 Pu\|^2_{(s)} + \|\varphi_1 u\|^2_{(\gamma)}\} \tag{12}$$

for any function $u \in D'(\Omega)$ such that $Pu \in H^{\mathrm{loc}}_{(s)}$, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$; $\varphi_1 \equiv 1$ in a neighborhood of $\operatorname{supp}\varphi$; $\gamma = \mathrm{const} < m+s-1$.

For any point (x,η) of the characteristic manifold N, consider the symmetric matrix

$$\mathfrak{A}(x,\eta) = \begin{pmatrix} p^{0(kj)}(x,\eta) & p^{0(k)}_{(j)}(x,\eta) \\ p^{0(j)}_{(k)}(x,\eta) & p^{0}_{(kj)}(x,\eta) \end{pmatrix} \quad (k,j = 1,\dots,n),$$

where for any multiindices α, β ($|\alpha| + |\beta| = 2$)

$$p^{\nu(\alpha)}_{(\beta)}(x,\xi) = \partial_x^\beta \partial_\xi^\alpha p^\nu(x,\xi).$$

By virtue of condition 1 the matrix $\mathfrak{A}(x,\eta)$ is positive semidefinite at any point $(x,\eta) \in N$. Let

$$\{Y_j = (a_1^j,\dots,a_n^j,b_1^j,\dots,b_n^j),\ j = 1,\dots,2n\}$$

be an orthonormal system of eigenvectors of the matrix $\mathfrak{A}(x,\eta)$, and let

$$\{Y_j^+,\ j \leq s(x,\eta)\}$$

be the set of eigenvectors corresponding to the positive eigenvalues of this matrix;

$$I(x,\eta) \equiv \left|p^{0(j)}_{(j)}(x,\eta) - ip^1(x,\eta)\right| + \sum_{j,k}^{S(x,\eta)} \left|\sum_{l=1}^n \left(a_l^j b_l^k - a_l^k b_l^j\right)\right|.$$

Theorem 4. A necessary and sufficient condition for the existence, for the operator P with conditions 1, 2, for every compact $K \subset \Omega$, of the estimate

$$\|u\|^2_{(m-1)} \leq C(K)\{\|Pu\|^2_{(0)} + \|u\|^2_{(0)}\}, \qquad u \in C_0^\infty(K), \tag{13}$$

is the fulfillment of the inequality

$$I(x,\eta) > 0 \quad \text{for every point } (x,\eta) \in N. \tag{14}$$

Theorem 5. If the pseudodifferential operator P satisfies conditions 1, 2 and (14), then for every compact $K \subset \Omega$ there exists a constant $C(K)$ such that the inequality

$$\|u\|^2_{(m-1)} + \sum_{s=1}^{n} \|P^{(s)}u\|^2_{(1/2)} + \|P_{(s)}u\|^2_{(-1/2)} \leq$$

$$\leq C(K)\{\|Pu\|^2_{(0)} + \|u\|^2_{(0)}\}, \qquad u \in C_0^\infty(K). \tag{15}$$

From estimate (15) (see, for example, [10]) it follows:

Theorem 6. If the differential operator P with infinitely differentiable coefficients satisfies conditions 1, 2 and (14), then for every compact $K \subset \Omega$ and every $s \in R^1$ there exists a constant $C(K, s)$ such that for any function $u \in D'(\Omega)$ such that $Pu \in H^{\text{loc}}_{(s)}$, the estimate

$$\|\varphi u\|^2_{(s+m-1)} \leq C(K, s)\{\|\varphi_1 Pu\|^2_{(s)} + \|\varphi_1 u\|^2_{(\gamma)}\},$$

holds, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$ and $\varphi_1 \equiv 1$ on $\operatorname{supp}\varphi$; $\gamma = \text{const} < s + m - 1$.

Theorem 7. A differential operator P satisfying conditions 1, 2 and (14) is hypoelliptic in Ω.

Remark. For a differential operator P of second order of the form (1), condition (14) is equivalent to the following inequality:

$$|X_0(x,\xi)| + \sum_{j,k=1}^{N} |[X_j, X_k](x,\xi)| > 0 \quad \text{for every point } (x,\xi) \in N.$$

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Note: Figure translations are in progress. See original paper for figures.