THE DIRICHLET PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

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Abstract

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THE DIRICHLET PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS DEPENDING ON A PARAMETER

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1. Preliminary remarks. Let, on $R_x^n \times R_\xi^n \times Q$ (Q is a certain angle with vertex at the origin on the complex plane), there be given a function $a_0(x, \xi, q)$, homogeneous of degree m in ξ and q, hereafter called the **symbol** of the operator A. The pseudodifferential operator A in R^n, canonically constructed from the symbol $a_0(x, \xi, q)$, is defined by the expression ([1])

$$Au \overset{\sim}{=} F_{\xi \to x}^{-1} a_0(x, \xi, q) \tilde{u}(\xi) = \mathrm{Op}(a(x, \xi, q)) u(x), \quad x \in R^n,$$

where $\tilde{u}(\xi) = F_{x \to \xi} u(x)$ is the Fourier transform of the function $u(x)$, and the second equality is a notation.

Let $\{\varphi_\nu, \Omega_\nu\}_0^M$ be a partition of unity of a domain G, bounded by a smooth $(n-1)$-dimensional surface Γ, such that $\Omega_0 \subset G$, $\Omega_\nu \cap \Gamma \neq \emptyset$ for $\nu = 1, 2, \dots, M$, and, in addition, $\{\varphi_\nu, \Omega_\nu\}_1^M$ gives a partition of unity of some neighborhood Γ_ρ of the boundary Γ. Each point $x \in \Gamma_\rho$ is uniquely determined by a pair (P', ρ), where ρ is the normal to Γ drawn through the point x ($\rho > 0$, if $x \in G$); P' is the point of intersection of the normal with Γ. As in ([2]), consider over Γ_ρ the tangent bundle of the form $T^*(\Gamma) \times R_\rho^1 \times R_\xi^1$, where $T^*(\Gamma)$ is the cotangent bundle of the manifold Γ, consisting of pairs (P', ζ'), ζ' being a tangent vector at the point P'; R_ρ^1 and R_ξ^1 are one-dimensional real spaces (mutually dual with respect to the Fourier transform). The symbol of the operator A, specified in Γ_ρ on the indicated tangent bundle, will be written in the form $a(P', \rho, \zeta', \zeta, q)$, and in each coordinate system x^ν ($x^\nu \in R^n, \nu$), associated with the neighborhood Ω_ν for $\nu = 1, 2, \dots, M$ (i.e., the axis x_n^ν coincides with the direction of the normal to Γ, and $x_n^\nu = 0$ is the equation of $\Omega_\nu \cap \Gamma$), in the form $a_\nu(x^\nu, \xi^{\nu'}, \xi_n^\nu, q)$, where $\xi^{\nu'} = \xi_1^\nu, \dots, \xi_{n-1}^\nu$. (Here we put $R^{n,0} = R^n$ and $x^0 = x$.) Taking into account that the expression Au in R^n was defined above, we shall also denote

$$Au = \mathrm{Op}(a(P', \rho, \zeta', \zeta, q)) u(P', \rho),$$

if $\operatorname{supp} u \subset \Gamma_\rho$.

As a result of the factorization ([3]) we shall have

$$a(P', \rho, \zeta', \zeta, q) = a_+(P', \rho, \zeta', \zeta, q) a_-(P', \rho, \zeta', \zeta, q),$$

where the function $a_+(P', \rho, \zeta', \zeta, q)$ in the ν-th coordinate system is a homogeneous function of ξ^ν and q of degree $\varkappa$, and admits an analytic continuation to the half-plane $\operatorname{Im} \xi_n^\nu > 0$; the function $a_-(P', \rho, \zeta', \zeta, q)$ has analogous properties with $\varkappa$ replaced by $m - \varkappa$ and $\operatorname{Im} \xi_n^\nu$ by $-\operatorname{Im} \xi_n^\nu$.

Below we use the Sobolev-Slobodetskii spaces of functions $H_s(G)$ and $H_r(\Gamma)$, as well as the space H_s^G of functions belonging to $H_s(G)$ and extended by zero outside the domain G. Along with the usual norms, in these spaces we use norms depending on a parameter. In the half-space R_+^n $(x_n > 0)$, for example, such a norm has the form

$$\|u_+\|_{s,q} = \|\Pi^+(|\xi_n - i\sqrt{|\xi'|^2 + |q|^2})^s lu(\xi))\|_0,$$

where $\operatorname{supp} u_+ \subset \overline{R}_+^n$; Π^+ is the Fourier image (in the generalized sense) of the operator of multiplication by the function $\theta^+(x_n)$, equal to zero for $x_n \le 0$ and to one for $x_n > 0$; $lu(x)$ is a smooth extension ([4]) of the function $u_+(x)$ to R^n. In the domain G an analogous norm

is defined in the known way by means of the norm in $R_+^{n,\nu}$ (and in R^n) and a partition of unity. The basic property of the indicated norm

$$\|u\|_{s-1,q}(G) \le (C/|q|)\|u\|_{s,q}(G)$$

(C does not depend on u or q) is established in the same way as in ([4]).

2. Existence and uniqueness theorem

Consider in $\overline{G}$ the Dirichlet problem

$$P(Au + Tu) = f(x), \qquad x \in G, \tag{1}$$

$$P_\Gamma(\partial^j u/\partial \rho^j) = g_j(P'), \qquad P' \in \Gamma, \quad j = 0, 1, \dots, \chi - 1, \tag{2}$$

where P is the restriction operator to G; $P_\Gamma v = Pv|_\Gamma$; T is the lower-order part of the operator; $u(x) = 0$ outside G.

We formulate the conditions imposed on the symbol of the operator A:

1. Homogeneity:

$$a_0(x, t\xi, tq) = t^m a_0(x, \xi, q), \qquad t > 0,$$

   m is an arbitrary complex number.

2. Smoothness and stabilization in x: the function $a_0(x, \xi, q)$ is infinitely differentiable in x and does not depend on x for sufficiently large values of $|x|$, uniformly with respect to ξ and q.
3. Smoothness in ξ and q: $D_x^p a_0(x, \xi, q)$

$$(D_x^p = (-i)^{|p|} \partial^{p_1}/\partial x_1^{p_1} \cdots \partial^{p_n}/\partial x_n^{p_n}, \quad p \text{ an arbitrary multi-index})$$

has continuous derivatives of any order with respect to ξ and q for $|\xi|+|q| \neq 0$.

4. Ellipticity (with parameter): for every x,

$$a_0(x, \xi, q) \neq 0$$

if ξ is real, $q \in Q$, and $|\xi| + |q| \neq 0$.

5. Smoothness in the domain (cf. 3): it is required that for all $\nu = 1, 2, \dots, M$, for any α and β (α a multi-index), the relations

$$\partial^\alpha_{\xi'_\nu} (\partial^\beta/\partial q^\beta) a_\nu(x^\nu, 0, 1, 0) = (-1)^{|\alpha|+\beta} e^{-i\pi m}\, \partial^\alpha_{\xi'_\nu} (\partial^\beta/\partial q^\beta) a_\nu(x^\nu, 0, -1, 0)$$

hold

$$(\partial^\alpha_{\xi'_\nu} = \partial^{\alpha_1}/\partial \xi_{\nu 1}^{\alpha_1} \cdots \partial^{\alpha_{n-1}}/\partial \xi_{\nu, n-1}^{\alpha_{n-1}}).$$

It follows from condition 5 that the factorization index χ is an integer. Below the case is considered in which, moreover, $\chi > 0$.

The following theorem is a generalization of the corresponding result of M. S. Agranovich and M. I. Vishik ([4]) on the solvability of elliptic boundary-value problems for differential equations.

Theorem 1. *Let conditions 1-5 be satisfied. Then for any functions* $f(x) \in H_{s-m}(G)$ *and* $g_j(P') \in H_{s-j-\frac{1}{2}}(\Gamma)$, *with* $s \geq \chi$, *and for sufficiently large values of* $|q|$, *there exists a unique solution* $u(x) \in H_s^G$ *of problem (1), (2), and, moreover, the a priori estimate*

$$\|u\|_{s,q}(G) \leq C \left(\|PAu\|_{s-m,q}(G) + \sum_{j=0}^{\chi-1} \|P_\Gamma(\partial^j u/\partial \rho^j)\|_{s-j-\frac{1}{2}}(\Gamma) \right) \tag{3}$$

is valid.

Let us make some remarks concerning the proof of this theorem. Writing $u(x)$ in the form of a sum

$$u(x) = u^{(1)}(x) + u^{(2)}(x),$$

defining $u^{(2)}(x)$ as the solution of the homogeneous polyharmonic equation

$$(\Delta^\chi + (-1)^\chi q^{2\chi})\, u^{(2)} = 0,$$

satisfying conditions (2), and taking into account that the assertion of the theorem (for $f(x) = 0$) for the function $u^{(2)}(x)$ was established in ([4]), we obtain that it is enough to prove the theorem for the Dirichlet problem with homogeneous boundary conditions:

$$PAu^{(1)} = f_1(x), \qquad x \in G, \qquad f_1(x) = f(x) - PAu^{(2)}(x); \tag{4}$$

$$P_\Gamma(\partial^j u^{(1)}/\partial\rho^j) = 0, \qquad j = 0, 1, \dots, \chi - 1. \tag{5}$$

The proof of the theorem for problem (4)-(5) in the half-space ($G = R^n_+$) is contained in ([5]). Applying known methods (see, for example, ([3,4])), it is not difficult to carry out the proof also for problems in a bounded domain; its basis consists in the construction of a regularizer and the use of the above property of norms depending on a parameter. We restrict ourselves

hereby indicating the form of the regularizer of the problem (4), (5):

$$Rf_1 = \psi_0(x)\,\mathrm{Op}\left(\frac{1}{a_0(x,\xi,q)}\right)\varphi_0(x) f_1(x) +$$

$$+\psi(\rho)\,\mathrm{Op}\left(\frac{1}{a_+(P',\rho,\xi',\xi,q)}\right)\chi(x)\,\mathrm{Op}\left(\frac{1}{a_-(P',\rho,\xi',\xi,q)}\right) f_2(P',\rho),$$

where $\psi_0(x) \in C_0^\infty(R^n_x)$, $\psi_0\varphi_0 \equiv \varphi_0$, $\psi(\rho) = \sum_{\nu=1}^M \varphi_\nu$, as already noted, is a function equal to unity in a neighborhood of Γ; $\chi(x)$ is equal to unity for $x \in G$ and zero for $x \in \bar{G}$; $f_2(P',\rho)$ is the continuation of the function $f_1(x)$ from $\Gamma_\rho \cap G$ to Γ_ρ.

3. Construction of the asymptotic expansion. This part of the note is, in its conceptual aspect, connected with the works of M. I. Vishik and L. A. Lyusternik (6) on the study of differential equations with a small parameter multiplying the highest derivatives.

Consider in $\bar{G}$ the problem

$$PA_\varepsilon u = f(x), \qquad x \in G; \tag{6}$$

$$P_\Gamma(\partial^j u/\partial\rho^j) = g_j(P'), \qquad P' \in \Gamma, \quad j = 0, 1, \dots, \varkappa - 1, \tag{7}$$

where $A_\varepsilon = \varepsilon^m A = \mathrm{Op}(a_0(x, \varepsilon\xi, \omega))$, $\varepsilon = |q|^{-1}$, $\omega = q/|q|$.

Suppose that conditions 1–5 are satisfied, as well as the following additional conditions:

6. Analyticity of the symbol in a strip: the function $a_\nu(x^\nu, \xi'^\nu, \xi^\nu_n, q)$ is analytic with respect to ξ^ν_n in the strip $-\lambda q_0 < \mathrm{Im}\,\xi^\nu_n < \mu q_0$, $\lambda > 0$, $\mu > 0$, $|q| \geq q_0 > 0$.

7. The function $f(x)$, the right-hand side of (6), vanishes on the boundary Γ together with derivatives of sufficiently high order.

Theorem 2. *Let conditions 1–7 be satisfied. Then, for sufficiently small values of ε, there exists an asymptotic expansion of the solution of problem (6), (7) of the form*

$$u(x,\varepsilon)=\sum_{l=0}^{N}\varepsilon^{l}\left(w_l(x)+\psi(\rho)v_l\left(P',\frac{\rho}{\varepsilon}\right)\right)+z_{N+1}(x,\varepsilon); \tag{8}$$

here the functions $w_l(x)$ and $v_l(P',\rho/\varepsilon)$ ($\psi(\rho)$ is defined above) are determined explicitly from two iteration processes corresponding to two different expansions in powers of ε of the symbol of the operator A_ε (see below), while for the remainder term $z_{N+1}(x,\varepsilon)$, for $s\geq 0$, the estimates

$$\|PD_x^s z_{N+1}\|_0(G)\leq C_s\varepsilon^{N+1-|s|}\left(\|f\|_{N_1+|s|}(G)+\sum_{j=0}^{\varkappa-1}|g_j|_{N_1,j+|s|}(\Gamma)\right),$$

hold, where N_1 and $N_{1,j}$ are certain numbers depending on $m,\varkappa$, and N, while C_s does not depend on ε and f.

The first iteration process is based on the direct expansion of the symbol $a_0(x,\varepsilon\xi,\omega)$ in powers of ε, which leads to the first splitting of the operator A_ε:

$$A_\varepsilon=a(x)+\sum_{j=1}^{N}\varepsilon^j A_j+\varepsilon^{N+1}R_{N+1}, \tag{9}$$

where $a(x)=a(x,0,\omega)$, $A_j=\sum_{|\alpha|=j}\frac{1}{\alpha!}\partial^\alpha a(x,0,\omega)D^\alpha$ (the derivatives ∂^α are taken with respect to the second argument), and R_{N+1} is a certain bounded pseudodifferential operator. If the solution of problem (6)–(7) is sought in the form $w_0(x)+\varepsilon w_1(x)+\cdots$, then from (6), (9) we obtain

$$w_0(x)=f(x),\qquad w_1(x)=-A_1w_0(x),\ldots,\quad w_l(x)=-\sum_{k=1}^{l}A_k w_{l-k}(x),$$

however, the sum $\sum_{l=0}^{N}\varepsilon^l w_l(x)$ will not, generally speaking, satisfy the boundary conditions (7). Therefore, making in (6) the substitution $\rho\to r=\rho/\varepsilon$ (with $\zeta\to\tau=\varepsilon\xi$) and expanding the symbol $a(P',\varepsilon r,\varepsilon\zeta,\tau,\omega)$ in ε, we obtain a second splitting of the operator A_ε of the form

$$A_\varepsilon=B_0+\sum_{j=1}^{N}\varepsilon^j B_j+\varepsilon^{N+1}R_{N+1,1}. \tag{10}$$

Unlike (9), all the operators in (10) are pseudodifferential; for example, $B_0 = \mathrm{Op}(a(P', 0, 0, \tau, \omega))$. Next, using (10), we seek a solution of the homogeneous equation corresponding to (6), with boundary conditions (7), in the form of the sum $v_0(P', r) + \varepsilon v_1(P', r) + \cdots$, which leads to a second iterative process for constructing the asymptotics. To determine $v_0(P', r)$ we obtain the following problem:

$$PB_0 v_0 = 0, \qquad P_\Gamma(\partial^j u/\partial\rho^j) = g_j(P'), \qquad j = 0, 1, \dots, \varkappa - 1.$$

The solution of this problem can be written explicitly if one takes into account that, upon passing to local coordinates, we obtain an equation with constant symbol (P' plays the role of a parameter) in the half-space ([3,5]). The remaining functions $v_l(P', r)$, $l = 1, 2, \dots, N$, are determined analogously, as solutions of nonhomogeneous equations with homogeneous boundary conditions. Since $v_l(P', r)$ have meaning only in Γ_ρ, in (8) the smoothing function $\psi(\rho)$ is used. Following (6), we shall call the functions $v_l(P', \rho/\varepsilon)$ functions of boundary-layer type. In the case under consideration, $v_l(P', \rho/\varepsilon)$ do not, generally speaking, have the form $C_k(P')(\rho/\varepsilon)^k \exp(-\lambda\rho/\varepsilon)$, $\mathrm{Re}\,\lambda > 0$ (as in (6)); however, they retain the property of exponential decay in ρ/ε, which is ensured by condition 6.

The estimate of the remainder $z_{N+1}(x, \varepsilon)$ in (8) is based on the a priori estimate (3) and on the properties of operators smooth in $\overline{G}$ (see condition 5).

We note that, in considering asymptotic methods, it apparently is not possible to weaken condition 7 (in contrast to the case of differential equations), as is indicated by the following simple example of an operator on the half-line with symbol of the form $a_\varepsilon(\xi) = i(\varepsilon\xi + i)^{-1}$. Indeed, we have

$$a_\varepsilon(\xi) = 1 + \varepsilon(i\xi) + \cdots + \varepsilon^N(i\xi)^N + \varepsilon^{N+1}(i\xi)^{N+1}(1 - i\varepsilon\xi)^{-1}.$$

Denoting $R_{N+1} = \mathrm{Op}((i\xi)^N(1 - i\varepsilon\xi)^{-1})$, we obtain that the quantity $\|R_{N+1}f\|_{L_2(0,\infty)}$ has, as $\varepsilon \to 0$, order $\varepsilon^{-N+k-1/2}$, if $f^{(i)}(0) = 0$ for $i = 0, 1, \dots, k-1$, and $f^{(k)}(0) \neq 0$.

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Note: Figure translations are in progress. See original paper for figures.