METRIC PROPERTIES OF MAPPINGS THAT LEAVE CERTAIN INTEGRAL FUNCTIONALS BOUNDED

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Abstract

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MATHEMATICS

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METRIC PROPERTIES OF MAPPINGS THAT LEAVE CERTAIN INTEGRAL FUNCTIONALS BOUNDED

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1. Let a continuous vector-function $y = f(x)$ with values in a set $\Delta \subset E^m$ be given in a domain D of n-dimensional Euclidean space E^n $(n \geq 2)$, where $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, $y_i = f_i(x)$ $(i = 1, 2, \dots, m)$. Suppose that this function leaves bounded the functional

$$I(f, D, F) = \int_D F\left(x, f, \frac{df}{dx}\right) dx,$$

where

$$\frac{df}{dx} = \left(\frac{\partial f_i}{\partial x_j}\right)$$

is a rectangular $n \times m$ matrix whose elements are the partial derivatives, understood in the sense of S. L. Sobolev, and $dx = dx_1 \dots dx_n$ is the volume element. With respect to $F(x, y, Z)$ we assume that it is a measurable function of its arguments, and moreover such that the inequality

$$F(x, y, Z) \geq h^n(x, y)\|Z\|^n$$

is satisfied, where $h(x, y)$ is a certain continuous nonnegative function defined in $D \times \Delta$; $Z = (z_{ij})$ is a rectangular $n \times m$ matrix;

$$\|Z\| = \left(\sum_{i=1}^{n}\sum_{j=1}^{m} z_{ij}^2\right)^{1/2}.$$

Consider on the set Δ a non-Euclidean metric with length element $ds = h\,dl$, where dl is the length element in E^m; $h = h(f^{-1}(y), y)$ in the case when $y = f(x)$

is a homeomorphism; if, however, $y = f(x)$ is not a homeomorphism, then we assume that $h(x, y)$ does not depend on x, and put $h = h(x, y)$. We shall call the metric thus introduced the metric h.

Let $\{S_r\}$ be a family of concentric spheres of radii r with center at some point $b \in E^n$: $S_r = \{x \in E^n : |x - b| = r\}$, $r_1 \leq r \leq r_2$, $r_1 \neq r_2$, and such that the sets $S'_r = S_r \cap D$ are nonempty for all $r \in [r_1, r_2]$. Suppose that on each set S'_r an open spherical disk K_r of spherical radius $R(r) \leq \pi r/2$ has been chosen. Let measurable nonnegative functions $\Omega(r)$ and $\beta(r)$ be defined on the segment $[r_1, r_2]$, such that $\Omega(r) \leq d_h(f(K_r))$, $\beta(r) \geq 2R(r)/\pi r$, where $d_h(G)$ is the diameter of the set $G \subset \Delta$ in the metric h.

Theorem. *Under the assumptions made, the inequality*

$$\int_{r_1}^{r_2} \frac{\Omega^n(r)}{r\beta(r)}\, dr \leq M_n I(f, D_{r_1 r_2}, F), \tag{1}$$

holds, where $D_{r_1 r_2} = \bigcup_{r_1 \leq r < r_2} S'_r$; M_n *is an absolute constant depending only on* n.

Proof is based on an inequality from paper ([1]) and is a generalization of the inequality of paper ([2]).

Inequality (1) makes it possible to obtain a number of important metric properties of mappings.

We shall use the following notation: $\rho(M_1, M_2)$ is the distance between sets in E^n; $|x' - x''|$ is the distance between points in E^n; $\overline{M}$ is the closure of the set M in E^n; ∂D is the boundary of the domain D in E^n; $d(M)$ is the diameter of the set M in E^n; $\rho_h(M_1, M_2)$ is the distance between sets in the metric h; $\widetilde{E}^n$ is the completion of n-dimensional space with respect to the spherical metric $\tilde{\rho}(x', x'')$; $\partial\widetilde{D}$ is the boundary of the domain D in $\widetilde{E}^n$.

Below we consider a family $\{f\}$ of homeomorphic mappings $y = f(x)$, defined in a domain $D \subset E^n$ and with values in a domain $\Delta \subset E^n$, $f(D) = \Delta_f \subset \Delta$ (in particular, it may be that $\Delta = E^n$). We assume that the boundary of the domain D is connected in $\widetilde{E}^n$.

To consider various normalizations of the family of functions $\{f\}$, we introduce the following classes of mappings. Suppose that a is some point of D, and that $M, \delta, \delta_1, \delta_2$ are arbitrary positive numbers.

Let $\infty \notin D$ (the domain D contains no exterior of any ball in E^n). We shall say that $\{f\} \subset A_1(a, M, \delta)$ if $|f(a)| \leq M$, $\rho(f(a), \partial\Delta_f) \leq \delta$ for all $f \in \{f\}$; $\{f\} \subset A_2(a, \delta)$ if $\rho(f(a), \partial\Delta_f) \geq \delta$ for all $f \in \{f\}$; $\{f\} \subset A_3(a, M, \delta_1, \delta_2)$ if $|f(a)| \leq M$, $\delta_1 \leq \rho(f(a), \partial\Delta_f) \leq \delta_2$ for all $f \in \{f\}$.

Let $\infty \in D$. In this case we shall assume that $f(\infty) = \infty$ and that the functions f are continuous in $\widetilde{E}^n$ at the point $x = \infty$. We shall say that $\{f\} \subset B_1(M, \delta)$

if $\rho(\partial\Delta_f, 0) \le M$ and $d(\partial\Delta_f) \le \delta$ for all $f \in \{f\}$; $\{f\} \subset B_2(\delta)$ if $d(\partial\Delta_f) \ge \delta$ for all $f \in \{f\}$; $\{f\} \subset B_3(M, \delta_1, \delta_2)$ if $\rho(\partial\Delta_f, 0) \le M$ and $\delta_1 \le d(\partial\Delta_f) \le \delta_2$ for all $f \in \{f\}$.

2. Order of growth. Let

$$I(f^{-1}, \Delta_f, \Phi) = \int_{\Delta_f} \Phi\left(y, f^{-1}, \frac{df^{-1}}{dy}\right) dy \le K, \tag{2}$$

where K does not depend on $f \in \{f\}$, and $\Phi(y, x, Z) \ge h^n(x)\|Z\|^n$, with $h(x)$ a nonnegative continuous function defined in D.

Let $\{f\} \subset A_1(a, M, \delta)$. Then for every connected compact set $G \subset D$ containing the point a, we shall have

$$|f|_G \le M + \delta \exp\left[M_{nK}\rho_h^{-n}(G, \partial D)\right], \tag{3}$$

where

$$|f|_G = \sup_{x\in G} |f(x)|.$$

Let now $\{f\} \subset B_1(M, \delta)$. Then for every bounded connected set $G \subset D$ such that the set $\partial D \cap \overline{G}$ is nonempty, the inequality

$$|f|_G \le M + \delta \exp\left[M_{nKq}h^{-n}(G, D)\right], \tag{4}$$

holds, where $q_h(G, D) = \inf d_h(S)$, and the infimum is taken over all Jordan surfaces $S \subset D$ such that ∂D is contained inside S and the set $S \cap G$ is nonempty.

Remark 1. Suppose that the set of zeros of the function $h(x)$ is compact in D. Then $\rho_h(G, \partial D) > 0$ for any G, and inequality (3) shows that the family $\{f\}$ is uniformly bounded on compact subsets of the domain D. Here $h(x)$ may tend to zero if $\rho(x, \partial D) \to 0$.

Suppose that the function $h(x)$ is positive in D, and let there exist a point $b \in \partial D$ and a positive number β such that, in some neighborhood of the point b, the inequality $h(x) \ge \beta$ holds. Then $q_h(G, D) > 0$ for any G, and inequality (4) gives uniform boundedness of the family $\{f\}$ on every set G.

Consider an example. Let $\Phi(y, x, Z) = h^n(x)\|Z\|^n$, where $h(x) = \dfrac{1}{1+|x|^2}$ defines the spherical metric in E^n. Let $\{f\}$ be a fami-

of Q-quasiconformal mappings. Then $I(f^{-1}, \Delta_f, \Phi) \le n^{n/2}Q^{n-1}\tilde{m}D$, where by $\tilde{m}D$ is denoted the spherical volume of the domain D. In this case we may put

$K = n^{n/2}Q^{n-1}\tilde{m}D$, and inequalities (3) and (4) give estimates for the order of growth for the family of mappings $\{f\}$.

3. Estimate from below of the distortion of the distance to the boundary. Let condition (2) be satisfied for the functions $f \in \{f\}$. If $\{f\} \subset A_2(a, \delta)$, then for every connected compact set $G \subset D$ containing the point a, the inequality

$$\rho(f(G), \partial\Delta_f) \geq \delta \exp[-M_{nK}\rho_h^{-n}(G, \partial D)]. \tag{5}$$

holds.

If $\{f\} \in B_2(\delta)$, then for every connected set $G \subset D$ such that $\rho(G, \partial D) > 0$, inequality (5) also holds.

4. Covering theorem. *Suppose that $\{f\} \subset A_2(a, \delta)$. Then for every closed domain $G \subset D$ such that $a \in G$, $\rho(a, \partial G) > 0$, we have*

$$\rho(f(a), \partial f(G)) \geq \delta \exp[-M_{nK}\varphi_h^{-n}(\rho(a, \partial G), D)],$$

where $\varphi_h(a, D) = \inf d_h(S)$, and the infimum is taken over all Jordan surfaces $S \subset D$ for which the point a lies inside S and $d(S) \geq \alpha$.

Remark 2. Let the function $h(x)$ be positive in D, and let there exist a point $b \in \partial D$ and positive numbers ε and β such that on the set $\{x \in D : \rho(x, b) \leq \varepsilon\}$ one has $h(x) \geq \beta$. Then on the interval $(0, d(D))$ the function $\varphi_h(a, D)$ is positive.

5. Equicontinuity. Let, for the family $\{f\}$, the condition

$$I(f, D, F) \leq K_1, \qquad I(f^{-1}, \Delta_f, \Phi) \leq K_2,$$

be satisfied, where $F(x, y, Z) \geq u^n(x, y)\|Z\|^n$, $\Phi(y, x, Z) \geq h^n(x)\|Z\|^n$, and the functions $u(x, y)$ and $h(x)$ are defined, continuous, and positive in $D \times \Delta$ and in D, respectively.

If $\{f\} \subset A_3(a, M, \delta_1, \delta_2)$, then for any two points $x', x'' \in G$ satisfying the condition

$$|x' - x''| < \rho_1/2, \quad \rho_1 = \rho(G, \partial D), \tag{6}$$

the inequality

$$|f(x') - f(x'')| \leq \frac{1}{m_1}[M_{nK}1]^{1/n} \ln^{-1/n} \frac{2\rho_1}{3|x' - x''|} \tag{7}$$

holds, where G is an arbitrary closed bounded domain from D, with $a \in G$, $\rho(a, \partial G) > 0$, and the set ∂G connected; $m_1 = \min_{x \in G_1, y \in H} u(x, y)$, $G_1 = \{x \in D : \rho(x, G) < \rho_1/2\}$, $H = \{y \in \Delta : \rho(y, \partial\Delta) > \alpha,\ |y| < \gamma\}$, $\alpha = \delta_1 \exp[-M_{nK} 2\rho_h^{-n}(G_1, \partial D)]$, $\gamma = M + \delta_2 \exp[M_{nK} 2\rho_h^{-n}(G_1, \partial D)]$.

If $\{f\} \subset B_3(M, \delta_1, \delta_2)$, then assume additionally that $h(x)$ satisfies the condition of the remark. Then inequality (7) holds under condition (6), if as the set G there is taken a bounded closed domain from D such that $D \setminus G$ consists of two components, one of which contains ∞ and the boundary of the second contains the set ∂D. In this case m_1 in (7) is defined as before, only another γ is taken, namely $\gamma = M + \delta_2 \exp[M_{nK} 2q_h^{-n}(G_2, D)]$, where $G_2 = D \setminus G_3$, and G_3 is the component of the set $D \setminus G$ containing ∞.

Corollary 1. *Under the conditions of Sec. 5, from the family of mappings $\{f\}$ one can choose a sequence which converges uniformly on every compact set of D to a continuous vector function defined in the domain D.*

6. Equiopenness. Suppose that the conditions of Sec. 5 are satisfied and, moreover, the function $h(x)$ satisfies the conditions of Remark 2. Then, in the case where $\{f\} \subset A_3(a, M, \delta_1, \delta_2)$, for any

two points x', $x'' \in G$, satisfying the condition

$$|x' - x''| < {}^2/_3 \rho_1 \exp\left\{-M_n K_1 \left[\frac{2}{m_1 \delta_1} \exp\left(\rho_h^{-n}(G, \partial G_1) + \varphi_h^{-n}(\rho(a, \partial G_1), D)\right)\right]^n\right\}; \tag{8}$$

the inequality

$$|f(x') - f(x'')| \geq {}^2/_3 \delta_1 \exp\left\{-M_{nK} 2\left[m_2^{-n} |x' - x''|^{-n} + \rho_h^{-n}(G, \partial G_1) + \varphi_h^{-n}(\rho(a, \partial G_1), D)\right]\right\}, \tag{9}$$

holds, where $m_2 = \min_{x \in G'} h(x)$, and the remaining notation is the same as in item 5.

In the case $\{f\} \subset B_3(M, \delta_1, \delta_2)$, inequality (9) will hold under condition (8), if in (8) and (9) the term $\varphi_h^{-n}(\rho(a, \partial G_1), D)$ is omitted.

Corollary 2. *Under the conditions of the present item, the family $\{f\}$ is uniformly open inside the domain D, and relations (8) and (9) give the order of this uniform openness.*

Remark 3. Under condition (8), together with inequality (9), inequality (7) is also satisfied, i.e., for the family $\{f\}$ there is a two-sided estimate of the distortion of distances inside the domain D.

Corollary 3. Let the conditions of item 6 be satisfied for the family $\{f\}$. Consider some sequence of the family $\{f\}$ converging at the point a to the value

a^*, if $\infty \notin D$, and an arbitrary sequence from $\{f\}$, if $\infty \in D$. Then from this sequence one can choose a subsequence $\{f^{(p)}\}$ $(p = 1, 2, ...)$ such that the sequence of domains $\{\Delta_{f^{(p)}}\}$ will converge to its kernel Δ_0 with respect to the point a^* (or with respect to the point $y = \infty$, if $\infty \in D$), and the sequence $\{f^{(p)}\}$ will converge uniformly inside D to a homeomorphic mapping $y = f(x)$ of the domain D onto the domain Δ_1, contained in Δ_0. Moreover, $\rho(f(a), \partial\Delta_1) \leq \delta_2$ (in the case $\infty \notin D$), and for the mapping $y = f(x)$ all inequalities that held for the family of mappings $\{f\}$ are satisfied, with the exception of inequality (5) in the case when $\infty \in D$, but $\infty \notin \Delta_1$.

Remark 4. Simple examples show that the domain Δ_1 need not coincide with Δ_0.

We give a sufficient condition for $\Delta_1 = \Delta_0$. Let $u(x, y) = u_1(x) \cdot u_2(y)$. Consider the set

$$E = \left\{ c \in \partial\bar{D} : \lim_{\rho(x,c)\to 0} u_1(x) = 0 \right\}.$$

Note that the set E is compact in $\widetilde{E}^n$. For the case $\infty \in D$ we shall assume that

$$\lim_{x\to\infty} u_1(x) > 0.$$

Let b be an arbitrary point of the set $\partial\bar{D} \setminus E$. If $b \neq \infty$, we shall assume that there exists $\varepsilon > 0$ such that, for any sphere S_r of radius $r < \varepsilon$ with center at the point b, every component of the set $S_r \cap D$ divides the domain D into two parts. If $b = \infty$, then let spheres of sufficiently large radius with center at the origin of coordinates have the analogous property. Suppose also that $\tilde{\rho}(f(x), \partial\Delta_f) \to 0$ uniformly with respect to $f \in \{f\}$ if $\tilde{\rho}(x, E) \to 0$. Then the domain Δ_1 coincides with the kernel Δ_0.

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Note: Figure translations are in progress. See original paper for figures.