REGULAR POINTS, THE SPECTRUM, AND EIGENFUNCTIONS OF NONLINEAR OPERATORS

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R. I. KACHUROVSKII

REGULAR POINTS, THE SPECTRUM, AND EIGENFUNCTIONS OF NONLINEAR OPERATORS

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The aim of the present note is to generalize a number of known facts from the spectral theory of linear operators to certain classes of nonlinear operators. In § 1 we study the location of regular points and points of the spectrum of nonlinear operators in Hilbert space; in § 2, the structure of the spectrum and of the set of regular values of completely continuous operators in Banach spaces. In § 3 a theorem is obtained on the existence of a continuum of eigenfunctions for the problem $\lambda F(u) = \Phi(u)$. In § 4 the results obtained are applied to nonlinear elliptic boundary-value problems.

Eigenfunctions and eigenvalues of nonlinear operators have previously been studied in a number of papers. For detailed information on them see ([1,2]).

Let E be a Banach space; $A : E \to E$ an operator, defined on all of E, linear or nonlinear. We recall the known definitions. A number λ is called a regular value of the operator A if the operator $R_\lambda = (A-\lambda I)^{-1} = A_\lambda^{-1}$ exists, is defined on all of E, and satisfies the Lipschitz condition

$$\|R_\lambda(x) - R_\lambda(y)\| \le K\|x-y\|, \quad \forall x,\ y \in E.$$

Here I is the identity operator in E, $K = \text{const}$. From the existence of R_λ it follows at once that $K > 0$. The set of all nonregular values in the complex plane is called the spectrum of the operator A. If $A(\theta) = \theta$, $A(x_\lambda) = \lambda x_\lambda$, $x_\lambda \ne \theta$, then the vector x_λ is called an eigenfunction of the operator A, corresponding to the eigenvalue λ. Obviously, eigenvalues are points of the spectrum. An operator A is called completely continuous if it is continuous and compact. A is called positively homogeneous of degree $\nu > 0$ if for all $x \in E$ and all $\mu \ge 0$ the equality

$$A(\mu x) = \mu^\nu A(x)$$

holds. For $\nu = 1$ the operator A is called positively homogeneous. If $A(\mu x) = \mu A(x)$, $\forall x \in E$ and all real μ, then A is called homogeneous. Let E^* be the space conjugate to E, and let $\Phi : E \to E^*$ be an operator defined on all of E. If there exists a functional $\varphi(x)$, defined on all of E, whose Fréchet derivative at every point $x \in E$ is equal to $\Phi(x)$, then Φ is called a potential operator, and φ a potential of the operator Φ. The operator $\Phi : E \to E^*$ is called strictly monotone in E if

$$(\Phi(x) - \Phi(y), x - y) > 0, \quad \forall x,\ y \in E,\ x \neq y.$$

Here (z, x) is the value of the functional $z \in E^*$ on the element $x \in E$.

1. Let H be a Hilbert space, and let $A : H \to H$ be an operator defined on all of H. Consider the numbers

$$m = \inf_{\substack{x,y \in H \\ x \neq y}} \frac{(A(x) - A(y), x - y)}{\|x - y\|^2}; \quad M = \sup_{\substack{x,y \in H \\ x \neq y}} \frac{(A(x) - A(y), x - y)}{\|x - y\|^2}. \quad (1)$$

Theorem 1. Let A be a continuous operator in H, and suppose the form $(A(x) - A(y), x - y)$ takes real values for all $x, y \in H$.

a) Every complex number $\lambda = \alpha + \beta i$, $\beta \neq 0$, is a regular value of the operator A; b) the spectrum of the operator A lies on the segment $[m, M]$;

c) if the numbers m, M in formulas (1) are finite and the operator A satisfies the Lipschitz condition

$$\|A(x) - A(y)\| \leq \max\{|m|, |M|\}\|x - y\|, \quad \forall x, y \in H,$$

then at least one of the numbers m, M is a point of the spectrum of the operator A.

Theorem 2. Let A be a completely continuous positively homogeneous potential operator of degree $\nu > 0$ in H, and let the form $(A(x), x)$ take real values for all $x \in H$.

a) If $\nu \neq 1$ and there exists an element $u_1 \in H$ such that $(A(u_1), u_1) > 0$, then every positive number is an eigenvalue of the operator A; b) if $\nu \neq 1$ and there exists an element $u_2 \in H$ such that $(A(u_2), u_2) < 0$, then every negative number is an eigenvalue of the operator A; c) if A is an even operator (i.e. $A(-u) = A(u)$, $\forall u \in H$, $A(x) \neq \theta$ in H and $\nu \neq 1$, then every real number $\gamma \neq 0$ is an eigenvalue of the operator A; d) if $\nu = 1$, then the numbers

$$m_0 = \inf_{\|x\|=1} (A(x), x),$$

$$M_0 = \sup_{\|x\|=1} (A(x), x)$$

are finite and at least one of them is an eigenvalue of the operator A.

The following results are also valid in Banach spaces.

2. Let E be a real Banach space, $A : E \to E$ an operator defined on all of E.

Lemma. Let A be a completely continuous homogeneous operator in E. In order that the number $\lambda \neq 0$ be a regular value of the operator A, it is necessary and sufficient that there exist a number $c > 0$ such that

$$\|A_\lambda(x) - A_\lambda(y)\| \geq c\|x - y\|, \quad \forall x, y \in E, \quad A_\lambda(x) = A(x) - \lambda x.$$

With the aid of the lemma one proves

Theorem 3. If zero is a point of the spectrum of a completely continuous homogeneous operator A in E, then the set of regular values of the operator A is an open set, and the spectrum is a closed set.

3. In this section E is a real reflexive Banach space, E^* its conjugate space, and $F : E \to E^*$, $\Phi : E \to E^*$ are operators defined on all of E. Consider the question of the existence of nonzero solutions (eigenfunctions) of the equation $\lambda F(x) = \Phi(x)$ with parameter λ.

Theorem 4 (cf. ([1], p. 343; ([2])). Let F be a strictly monotone potential operator with potential f, $F(\theta) = \theta$,

$$\lim_{\|x\| \to \infty} f(x) = +\infty.$$

Let Φ be a completely continuous potential operator, $\Phi(\theta) = \theta$. Then in the space E there exists a continuum of distinct nonzero solutions of the equation

$$\lambda F(x) = \Phi(x)$$

(these solutions may correspond to one or to different values of the parameter λ).

If, in addition to the conditions of the theorem, the operators F and Φ are positively homogeneous of degree $\gamma > 0$, then the numbers

$$\tilde{m}_0 = \inf_{x \in X} (\Phi(x), x), \quad \tilde{M}_0 =$$

$$= \sup_{x \in X} (\Phi(x), x)$$

are finite, and for λ equal to at least one of them the problem

$$\lambda F(x) = \Phi(x)$$

has a nonzero solution. Here

$$X = \{x : (F(x), x) = 1,\ x \in E\}.$$

4. We shall dwell on some applications of the results obtained above. We restrict ourselves here only to the application of Theorems 2 and 4 to the study of the existence of eigenfunctions and eigenvalues in nonlinear elliptic boundary-value problems.

Let Ω be a bounded domain of the n-dimensional space R^n with piecewise smooth boundary $\partial\Omega$, $1 < p < \infty$. On the Sobolev space $\overset{\circ}{W}{}_p^m(\Omega)$ consider the functionals

$$f(u) = \int_\Omega \psi(x, u, \dots, D^m u)\, dx, \quad \varphi(u) = \int_\Omega \omega(x, u, \dots, D^{m-1} u)\, dx. \tag{2}$$

We shall say that the function $\delta(x, u, \dots, D^\alpha u, \dots, D^k u)$ satisfies the Carathéodory conditions if, for almost all $x \in \Omega$, it is continuous jointly in the variables $D^\alpha u$ ($|\alpha| \le k$) and is measurable in x in Ω for any

values $D^\alpha u$. Denote the partial derivatives of the function ψ with respect to $D^\alpha u$ by ψ_α, and those of the function ω with respect to $D^\sigma u$ by ω_σ.

Condition I. There exist $\psi_\alpha, \omega_\sigma$ for all α, σ, $|\alpha| \le m$, $|\sigma| \le m-1$, and the functions $\psi, \omega, \psi_\alpha, \omega_\sigma$ satisfy the Carathéodory conditions and the inequalities

$$|\psi(x, u, \dots, D^m u)| \le c \left[k_1(x) + \sum_{|\gamma| \le m} |D^\gamma u|^{p_\gamma} \right];$$

$$\left|\omega(x, u, \dots, D^{m-1} u)\right| \le c \left[k_1(x) + \sum_{|\beta| \le m-1} |D^\beta u|^{q_\beta} \right];$$

$$|\psi_\alpha(x, u, \dots, D^m u)| \le c \left[k_2(x) + \sum_{|\gamma| \le m} |D^\gamma u|^{p_{\alpha\gamma}} \right];$$

$$\left|\omega_\sigma(x, u, \dots, D^{m-1} u)\right| \le c \left[k_2(x) + \sum_{|\beta| \le m-1} |D^\beta u|^{p_{\sigma\beta}} \right],$$

where $k_1(x) \in L_1(\Omega)$; $k_2(x) \in L_{p'}(\Omega)$; $1/p + 1/p' = 1$; $c = \text{const}$; $0 \le p_\gamma \le np/[n - (m - |\gamma|)p]$; $0 \le p_{\alpha\gamma} \le [n(p-1) + (m - |\alpha|)p]/[n - (m - |\gamma|)p]$, if $n > (m - |\gamma|)p$; $p_\gamma \ge 0$, $p_{\alpha\gamma} \ge 0$ are arbitrary numbers if $n \le (m - |\gamma|)p$; $0 \le q_\beta < np/[n - (m - |\beta|)p]$, $0 \le p_{\sigma\beta} \le [n(p-1) + (m - |\sigma|)p]/[n - (m - |\beta|)p]$, if $n > (m - |\beta|)p$; $q_\beta \ge 0$, $p_{\sigma\beta} \ge 0$ are arbitrary numbers if $n \le (m - |\beta|)p$.

Condition II. For any $u, v \in \dot{W}_p^m(\Omega)$, $u \ne v$, the inequality

$$\int_\Omega \sum_{|\alpha| \le m} [\psi_\alpha(x, u, \dots, D^m u) - \psi_\alpha(x, v, \dots, D^m v)]\, D^\alpha(u - v)\, dx > 0$$

holds.

In the space $\dot{W}_p^m(\Omega)$ let us consider the question of the existence of nonzero solutions (eigenfunctions) in the nonlinear elliptic boundary-value problem with parameter λ

$$\lambda \sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \psi_\alpha(x, u, \dots, D^m u) = \sum_{|\sigma|\le m-1} (-1)^{|\sigma|} D^\sigma \omega_\sigma(x, u, \dots, D^{m-1} u)$$

$$D^i u\big|_{\partial\Omega} = 0, \qquad |i| \le m-1. \tag{3}$$

Theorem 5. Suppose that Conditions I, II hold and

$$\sum_{|\alpha|\le m} (-1)^{|\alpha|} D^\alpha \psi_\alpha(x, u, \dots, D^m u)\big|_{u=0} =$$

$$= \sum_{|\sigma|\le m-1} (-1)^{|\sigma|} D^\sigma \omega_\sigma(x, u, \dots, D^{m-1} u)\big|_{u=0} = 0$$

in the space $W_{p'}^m(\Omega)$. Let the functional $f(u)$, defined by equalities (2), be such that $\lim_{\|u\|\to\infty} f(u) = +\infty$, $\|u\|_1 = \|u\|_{\dot{W}_p^m(\Omega)}$. Then in the space $\dot{W}_p^m(\Omega)$ there exists a continuum of distinct nonzero solutions of problem (3), and these solutions may correspond to one and the same or to different values of the parameter λ.

If, in addition to the assumptions of the theorem, the functionals f, φ, defined by equalities (2), satisfy the conditions $f(\mu u) = \mu^{\nu+1} f(u)$, $\varphi(\mu u) = \mu^{\nu+1}\varphi(u)$, $\nu > 0$, for all $u \in \dot{W}_p^m(\Omega)$ and all $\mu \ge 0$, then the numbers

$$\widetilde{m}_0 = \inf_{u\in X} \sum_{|\sigma|\le m-1} \int_\Omega \omega_\sigma(x, u, \dots, D^{m-1}u) D^\sigma u \, dx,$$

$$\widetilde{M}_0 = \sup_{u\in X} \sum_{|\sigma|\le m-1} \int_\Omega \omega_\sigma(x, u, \dots, D^{m-1}u) D^\sigma u \, dx$$

are finite, and for λ equal to at least one of them, problem (3) has nonzero solutions. Here

$$X = \left\{ u : \ u \in \dot{W}_p^m(\Omega), \quad \sum_{|\alpha|\le m} \int_\Omega \psi_\alpha(x, u, \dots, D^m u) D^\alpha u \, dx = 1 \right\}.$$

We now give an application of Theorem 2. Consider the following conditions.

Condition III. The functions $a_{\alpha\delta}(x)$, defined in Ω, are such that the form

$$a(u,v)=\sum_{|\alpha|,|\delta|\le m}\int_\Omega a_{\alpha\delta}D^\alpha u D^\delta v\,dx$$

is symmetric on $\dot W_2^m(\Omega)$ and satisfies the inequalities

$$\mu_0'\|u\|_1^2\ge a(u,u)\ge \mu_0\|u\|_1^2,\quad \forall u\in \dot W_2^m(\Omega),$$

where μ_0', μ_0 are constants, $\mu_0>0$, $\|u\|_1=\|u\|_{\dot W_2^m(\Omega)}$.

Condition IV. The functions ω_σ exist for all σ, $|\sigma|\le m-1$, $\omega_\sigma=\partial\omega/\partial z_\sigma$, $z_\sigma=D^\sigma u$, and moreover ω, ω_σ satisfy the Carathéodory conditions and the inequalities

$$|\omega(x,u,\dots,D^{m-1}u)|\le c\left[k_1(x)+\sum_{|\beta|\le m-1}|D^\beta u|^{q_\beta}\right];$$

$$|\omega_\sigma(x,u,\dots,D^{m-1}u)|\le c\left[k_2(x)+\sum_{|\beta|\le m-1}|D^\beta u|^{p_{\sigma\beta}}\right],$$

where $k_1(x)\in L_1(\Omega)$, $k_2(x)\in L_2(\Omega)$, $0\le q_\beta<2n/[\,n-2(m-|\beta|)\,]$, $0\le p_{\sigma\beta}\le [\,n+2(m-|\sigma|)\,]/[\,n-2(m-|\beta|)\,]$, if $|\beta|>m-n/2$; $q_\beta\ge 0$, $p_{\sigma\beta}\ge 0$ are arbitrary numbers if $|\beta|\le m-n/2$.

Condition V. The functional

$$\varphi(u)=\int_\Omega \omega(x,u,\dots,D^{m-1}u)\,dx$$

takes only real values and satisfies the condition

$$\varphi(\mu u)=\mu^{\nu+1}\varphi(u),\quad \forall u\in\dot W_2^m(\Omega),\quad \forall\mu\ge 0.$$

Here the number ν is fixed, $\nu>0$, $\nu\ne 1$.

Consider the boundary-value problem with parameter λ

$$\lambda\sum_{|\alpha|,|\delta|\le m}(-1)^{|\delta|}D^\delta(a_{\alpha\delta}(x)D^\delta u)=\sum_{|\sigma|\le m-1}(-1)^{|\sigma|}D^\sigma\omega_\sigma(x,u,\dots,D^{m-1}u),\quad (4)$$

$$D^i u\Big|_{\partial\Omega}=0,\quad |i|\le m-1.$$

We shall seek nonzero solutions of problem (4) in the space $\dot W_2^m(\Omega)$.

Theorem 6. Suppose Conditions III–V are satisfied.

a) If there exists an element $u_1 \in \dot{W}_2^m(\Omega)$ such that $\varphi(u_1) > 0$, then for every $\lambda > 0$ problem (4) has a nonzero solution; b) if there exists an element $u_2 \in \dot{W}_2^m(\Omega)$ such that $\varphi(u_2) < 0$, then for every $\lambda < 0$ problem (4) has a nonzero solution; c) if $\varphi(-u) = -\varphi(u)$, $\forall u \in \dot{W}_2^m(\Omega)$, and the functional $\varphi(u) \not\equiv 0$ in the space $\dot{W}_2^m(\Omega)$, then for every $\lambda \neq 0$ problem (4) has a nonzero solution.

Let us also note that under the conditions of Theorem 6 we have

$$\varphi(u) = \frac{1}{\nu+1} \sum_{|\sigma| \le m-1} \int_\Omega \omega_\sigma(x, u, \dots, D^{m-1}u) D^\sigma u \, dx \quad \text{for all } u \in \dot{W}_2^m(\Omega).$$

The simplest example of problem (4) is the problem, studied in a number of works,

$$\Delta u = u^2, \quad u|_{\partial\Omega} = 0.$$

All-Union Correspondence Electrotechnical
Institute of Communications
Moscow

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.