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Abstract

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MATHEMATICS

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TOPOLOGICAL EMBEDDINGS IN E^3 OF SIMPLE ARCS AND CLOSED CONTOURS

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1. In ([7,8]) it was proved that for every two-dimensional manifold M^2 in Euclidean space E^3 (or in a piecewise-linear three-dimensional manifold M^3) and number $\varepsilon > 0$ there exist a polyhedral manifold $N^2 \subset E^3$, homeomorphic to M^2, and an ε-pseudoisotopy $F_t,\ 0 \le t \le 1$, of the space E^3, taking N^2 onto M^2. (A pseudoisotopy is a homotopy such that F_t is a homeomorphism for every $t < 1$.) F_t is an ε-pseudoisotopy if, for all x and t, $\rho(x, F_t(x)) < \varepsilon$, where ρ is distance. Moreover, $F_0 = 1$, $F_1(N^2) = M^2$, F_{1/N^2} is a homeomorphism, and the set of nondegenerate inverse images of the mapping F_1 is zero-dimensional and is contained in the set of wild points of M^2.

Krups extended this theorem to embeddings in a piecewise-linear three-dimensional manifold with boundary of a two-dimensional polyhedron having no locally separating points ([5]). It remained unknown whether an analogous pseudoisotopy exists for embeddings in E^3 of one-dimensional polyhedra, in particular simple arcs. It is plausible (although not proved) that for an arbitrary simple arc $l \subset E^3$ one cannot construct a pseudoisotopy of the space E^3 taking a line segment onto l. We give a sufficient condition for the existence of a pseudoisotopy $F_t : E^3 \to E^3$ taking a line segment onto a given simple arc and some polyhedral closed contour onto a given one.

It is known ([6,2,9]) that in order for a simple arc (closed contour) l in E^3 to be tame, i.e., for there to exist a homeomorphic mapping of E^3 onto itself taking l onto a polyhedron, it is necessary and sufficient that two conditions be fulfilled:

1) **local peripheral unknottedness**, i.e., for every point $x \in l$ there exists a three-dimensional cell Q_x such that $x \in \operatorname{int} Q_x$ and $\partial Q_x \cap l^*$ consists of two points (or one, if x is an end point);

2) **local unknottedness**, i.e., for every point $x \in l$ there exists a neighborhood l_x of the point x on l, which lies on a disk (two-dimensional cell):

$x \in l_x \subset d_x$.

There exist examples of wild simple arcs in E^3 for which one of these conditions is fulfilled ([1]).

2. We prove the following theorems:

Theorem 1. *For every locally unknotted simple arc l in E^3 there exists a pseudoisotopy $F_t : E^3 \to E^3$ from the identity mapping, taking a line segment I^1 onto l. Moreover F_{1/I^1} is a homeomorphic mapping of the segment I^1 onto the arc l, and the set of points $x \in E^3$ whose inverse images under the continuous mapping F_1 are nondegenerate is zero-dimensional and is contained in the set of wild points of the arc l.*

If a homeomorphic mapping $\varphi : I^1 \to l$ is given, then the pseudoisotopy F_t can be constructed so that $F_{1/I^1} = \varphi$.

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* By int M we denote the interior of the manifold M, and by ∂M its boundary.

Theorem 2. For every locally unknotted closed contour $C \subset E^3$ and number $\varepsilon > 0$ there exists a polyhedral closed contour P and an ε-pseudoisotopy $F_t : E^3 \to E^3$ carrying P onto C, with $F_0 = 1$, F_1/P a homeomorphism, $F_1(P) = C$, and the set of nondegenerate point-inverses of the mapping F_1 zero-dimensional and contained in the set of wild points of the contour C.

Both of these theorems follow easily from the theorem on pseudoisotopy for two-dimensional manifolds with boundary ([5]) in E^3, and from the following theorem and its corollary.

Theorem 3. Every locally unknotted simple arc l in E^3 lies on a disk $D \subset E^3$, the set of wild points of which coincides with the set of wild points of l.

Corollary. Every locally unknotted simple closed contour C in E^3 lies on a two-dimensional manifold with boundary $M^2 \subset E^3$, and the set of wild points of M^2 coincides with the set of wild points of C.

The proof of Theorem 3 contains many technical details. We shall briefly set forth its idea. A detailed text will be published later.

Let l be a locally unknotted simple arc in E^3 with endpoints a and b. Then l can be represented as a sum of arcs l_i:

$$l = \bigcup_{i=1}^{n} l_i, \qquad l_1 \ni a, \qquad l_n \ni b; \tag{1}$$

$$l_i \cap l_j = \Lambda, \qquad \text{if } |i-j| > 1;$$

$l_i \cap l_{i+1}$ is an interval l_i', and l_i lie on disks $l_i \subset d_i$. The disks d_i may be chosen so that

$$d_i \cap l = l_i, \qquad i \leq n; \qquad d_i \cap d_j = \Lambda, \qquad \text{if } |i-j| > 1,$$

and d_i is contained in a neighborhood U_i of the arc l_i, with $U_i \cap U_j = \Lambda$, $|i-j| > 1$. Moreover, l_i lies on the boundary of d_i, and by Bing' s approximation theorem ([3]) the disks d_i may be assumed locally polyhedral at the points $d_i \setminus l$.

We construct successively disks D_k so that $D_k \cap l = \bigcup_{i=1}^{k} l_i$ and $D_k \setminus l$ is locally polyhedral. Put $D_1 = d_1$, and suppose that a disk D_k, $1 \leq k < n$, has been constructed. Obviously $D_k \cap d_{k+1} \neq \Lambda$, since $D_k \cap d_{k+1} \supset l_k \cap l_{k+1} = l'_k$.

Consider the different cases.

1) $D_k \cap d_{k+1} = l'_k$. Then put $D_{k+1} = D_k \cup d_{k+1}$.

2) $D_k \cap d_{k+1} \supset L$, where L is either a simple arc, one end of which lies on l'_k, or a continuum homeomorphic to the closure of the graph of the function $\sin 1/x$ with its continuum of condensation on l'_k, and L separates both D_k and d_{k+1}:

$$D_k \setminus L = D'_k \cup D''_k, \qquad d_{k+1} \setminus L = d'_{k+1} \cup d''_{k+1},$$

where $D'_k \ni a$ and $d''_{k+1} \supset l_{k+1} \setminus l'_k$, and $D'_k \cap d''_k = L$. Then we obtain:

$$D_{k+1} = \overline{D}'_k \cup \overline{d}''_{k+1} * . \tag{2}$$

In the general case the intersection $D_k \cap d_{k+1}$ may be much more complicated. We transform the disks d_{k+1} (and sometimes D_k) inside $U_k \cap U_{k+1}$ so as to obtain 1) or 2). By a small displacement of d_{k+1}, fixed on l_{k+1}, the intersection $D_k \cap (d_{k+1} \setminus l)$ is brought into general position. Since $D_k \setminus l$ and $d_{k+1} \setminus l$ are polyhedra, $D_k \cap d_{k+1}$, apart from l'_k and L_i, if 1) or 2) is not satisfied, may contain, possibly intersecting, polyhedral closed contours γ_i and polygonal lines λ_i, both of whose ends lie on ∂d_{k+1} or on ∂D_k, as well as polygonal lines μ_i, having one end on $\partial d_{k+1} \setminus l$ and the other on $\partial D_k \setminus l$. The number of contours γ_i and polygonal lines λ_i, μ_i outside an arbitrary neighborhood of l'_k is finite. By rebuilding d_{k+1} and D_k inside $U_k \cap U_{k+1}$, the contours γ_i and polygonal lines λ_i are successively eliminated. If, after eliminating all

$*$ $\overline{A}$ denotes the closure of the set A.

γ_i and λ_i and a finite number of μ_i we do not obtain 1) and 2), then the transformed intersection $D_k \cap (d_{k+1} \setminus l)$ consists of an infinite number of nonintersecting polygonal lines μ_i tending to l'_k as i increases. The ends of μ_i form two sequences of points $\{x_i\} \to x_0$ and $\{x'_i\} \to x'_0$, converging to the ends of l'_k. Joining, by a polygonal line in the strip $D_k \cap \overline{U}_k \cap \overline{U}_{k+1}$, between two neighboring μ_i (and between μ_1 and ∂D_k) the points of the sequence $\{x_i\}$, we construct on

D_k a simple arc τ with end $x_0 = l'_k \cap \overline{(D_k \setminus l'_k)}$. Similarly, we construct on d_{k+1} a simple arc ν with end x'_0. The arc τ cuts off from D_k a disk $\widetilde{D}_k \subset U_k \cap U_{k+1}$, and the arc ν cuts off from d_{k+1} a disk $\tilde{d}_{k+1} \subset U_k \cap U_{k+1}$. The polygonal lines μ_i divide $\widetilde{D}_k$ and $\tilde{d}_{k+1}$ into strips $\sigma_i \subset \widetilde{D}_k$ and $s_i \subset \tilde{d}_{k+1}$, bounded by neighboring polygonal lines μ_i and μ_{i+1} on $\widetilde{D}_k$ (on $\tilde{d}_{k+1}$, other μ_j may lie between μ_i and μ_{i+1}), and $\sigma_i \cup s_i = \pi_i$ is a cylinder with generator μ_i and base $\tau_i \cup \tau'_i$, where τ_i is the segment of τ between μ_i and μ_{i+1}, and τ'_i is the segment of $\partial d_{k+1} \setminus l$ from x_i to x_{i+1}; τ'_i may intersect τ at some further points x_j. We choose on the cylinders π_i new generators θ_i, lying on $\pi_i \setminus \partial\pi_i$, so that the diameters of θ_i tend to zero together with $1/i$. We prove that:

A. *For sufficiently large i, the contour θ_i bounds a disk contained in $U_k \cap U_{k+1}$ and intersecting π_i in θ_i.*

Replacing, if necessary, the disk d_{k+1} by a smaller disk not containing those μ_i for which θ_i does not bound a disk in $U_k \cap U_{k+1}$, we may assume that A is true for all i. The diameters of the disks spanned by θ_i tend to zero together with $1/i$.

Two cases are possible:

a) θ_i bounds in $U_k \cap U_{k+1}$ a disk not intersecting l.

b) Every disk in $U_k \cap U_{k+1}$ bounded by θ_i intersects l.

In case a) we rearrange each strip s_i cut off from $\tilde{d}_{k+1}$ by the polygonal lines μ_i and μ_{i+1} in such a way that all $\mu_j \subset s_i$ are replaced by several polygonal lines of the form λ_p, the ends of which lie on $\partial D_k \setminus l$; moreover, the new strip s'_i contains a disk spanned by θ_i. After, under such a rearrangement of d_{k+1}, all μ_i satisfying a) have been successively replaced by arcs λ_p, we eliminate the λ_p, rearranging D_k in the part $\widetilde{D}_k$.

When all μ_i for which a) holds have been eliminated, the intersection $D_k \cap d_{k+1} \setminus l$ (we keep the former notation) consists of μ_i for which b) holds. If there is a finite number of such μ_i, then it remains to trim the disk d_{k+1} in order to obtain 1). Let $D_k \cap d_{k+1}$ contain an infinite set of μ_i. It is proved that the order of the μ_i on D_k and on d_{k+1} is the same and $s_i \cap \mu_j = \Lambda$, if $i \neq j \neq i+1$. Replacing the cylinder π_i, if necessary, by the cylinder π'_i cut off from π_i by a pair of closed contours close to $\partial\pi_i$, we show that each π'_i lies on a sphere S_i, close to l'_k for large i, and moreover

$$\pi'_i \cap \pi'_{i+1} \subset \mu_{i+1}; \qquad \pi'_r \subset \operatorname{int} S_i, \quad r \geq i+2,$$

$$S_i \cap S_{i+1} \subset \mu_{i+1}; \qquad \operatorname{int} S_{i+1} \subset \operatorname{int} S_i;$$

and

$$\bigcap_{i=1}^{\infty} \operatorname{int} S_i = \tilde{l}_k$$

is the segment l'_k. Applying the Bing-Kirkor criterion ([4]), one can show that the segment $\tilde{l}_k$ is tame. Next we transform the disk d_{k+1}, changing it in the part $\tilde{d}_{k+1}$ so that each polygonal line μ_i is replaced by a simple arc μ'_i, one end of which lies at the end y_0 of the arc $\tilde{l}_k$ near x_0 and which is locally polyhedral, except for the point y_0. After this transformation, the intersection $D_k \cap d_{k+1}$ is a bundle of arcs $\tilde{l}_k$ and μ'_i, $i = 1, 2, ...$, issuing from the point y_0, with $\mu'_i \cap \mu'_j = y_0$, $i \neq j$, and μ'_{i+1} lying on D_k and on d_{k+1} between μ'_i and $\tilde{l}_k$.

Consequently, for $D_k \cap d'_{k+1}$ case 2) holds, and D_{k+1} is defined by formula (2).

Thus the disks D_k are successively constructed, and finally $D = D_n \supset l$. If C'_1 is a locally unknotted closed contour, then the arc

$l' = C \setminus (l_n \setminus l_{n-1})$ is locally unknotted and lies on the disk D. In order to obtain a manifold with boundary M^2 containing the contour C, it suffices, starting from the disks $D \supset l'$ and $d_n \supset l_n$, to carry out the construction described above in neighborhoods of both components $l'_n \cap l_n$.

The same method proves

Theorem 4. Let P be a topological polyhedron in E^3, all of whose one-dimensional simplexes are locally unknotted, and at each vertex x the following condition is satisfied:

C. In E^3 there exists a neighborhood U_x of the vertex x such that a sufficiently fine triangulation of $P \cap \overline{U}_x$ is a subpolyhedron of a polyhedron $\pi_x \subset \overline{U}_x$ having no locally separating points.

Then some triangulation of the polyhedron P is a subpolyhedron of a polyhedron $\Pi \subset E^3$ without locally separating points.

The polyhedron Π satisfies the hypotheses of Craggs' theorem on pseudoisotopy ([5]); consequently Theorem 2 is true for Π, and hence also for P.

Question 1. Is the condition of local unknottedness of a simple arc and of a simple closed contour **necessary** for the existence of a pseudoisotopy satisfying the hypotheses of Theorems 1 and 2?

Question 2. Are condition C and the local unknottedness of the one-dimensional simplexes of the polyhedron P necessary for the existence of an analogous pseudoisotopy?

Theorem 3, and consequently Theorems 1, 2, and 4, remain valid for the embedding of a locally unknotted simple arc (closed contour) in a piecewise-linear three-dimensional manifold.

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Note: Figure translations are in progress. See original paper for figures.