CONTINUITY OF THE METRIC PROJECTION AND SOME GEOMETRIC PROPERTIES OF THE UNIT SPHERE IN A BANACH SPACE

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 513.88:513.83

MATHEMATICS

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CONTINUITY OF THE METRIC PROJECTION AND SOME GEOMETRIC PROPERTIES OF THE UNIT SPHERE IN A BANACH SPACE

(Presented by Academician A. N. Tikhonov, 1 VII 1968)

Let E and F be metric spaces: 2^F is the set of all closed subsets of the space F. A mapping $\varphi : E \to 2^F$ is called a multivalued mapping from E into F. φ is called upper semicontinuous ([3]) if the set $\{x \in E : \varphi x \subset G\}$ is open in E for every open subset $G \subset F$. We shall call the mapping φ H-upper semicontinuous (or upper semicontinuous in the Hausdorff sense) if $x_n \to x$ implies

$$\rho(\varphi x_n, \varphi x) = \sup\{\rho(y, \varphi x) : y \in \varphi x_n\} \to 0^*.$$

Both of these definitions coincide with the usual definition of continuity if φ is single-valued.

Let M be a set in E. Denote by T_M the multivalued mapping $E \to 2^M$ which assigns to each point $x \in E$ the set

$$T_M x = \{y \in M : \rho(x, y) = \rho(x, M)\};$$

T_M is called the metric projection of E onto M ([2]). Everywhere in what follows: X is a B-space over the field of real numbers; $S = \{x \in X : \|x\| = 1\}$, $S^* = \{f \in X^* : \|f\| = 1\}$; $x_n \rightharpoonup x_0$, if the sequence $\{x_n\}$ converges weakly to x_0; $f_n \rightharpoonup f_0$, if $f_n(x) \to f_0(x)$ for every $x \in X$.

Definition 1. We shall say that X satisfies condition (EPS) if from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\} \subset S^*$, $f_0 \in S^*$ $(f_0 \neq f_n,\ n = 1, 2, ...)$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $f_n \rightharpoonup f_0$, and $\rho(x_0, H_0 \cap H_n) \to 0$, $\rho(x_n, H_0 \cap H_n) \to 0$, where $H_0 = \{x : f_0(x) = 1\}$, $H_n = \{x : f_n(x) = 1\}$, it follows that $x_n \to x_0$.

Definition 2. We shall say that X satisfies condition (CR) (compactly rotund) (respectively (WCR) (weakly compactly rotund)) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or compact (respectively, it is either empty or weakly compact).

Definition 3. We shall say that X satisfies condition (HR) (H-rotund) (respectively (WHR) (weakly H-rotund)) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or satisfies the following condition: from $f(y_n) = 1$, $\rho(y_n, \{x : f(x) = 1\} \cap S) \to 0$ it follows that

$$\rho(y_n, \overline{\text{co}}\{y_n\} \cap S) \to 0^{**}$$

(respectively, it is either empty, or satisfies the following condition: from $f(y_n) = 1$, $\rho(y_n, \{x : f(x) = 1\} \cap S) \to 0$ and the weak compactness of $\{y_n\}$, it follows that $\rho(y_n, \overline{\text{co}}\{y_n\} \cap S) \to 0$).

If (A) and (B) are some properties of the space X, then by (A) $\wedge$ (B) we shall denote the property of the space X which consists in the fact that X possesses both property (A) and property (B).

Theorem 1. *In order that, in a reflexive B-space X, the metric projection onto every convex closed set be H-semi-*

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* By definition, we assume that $\rho(y, \Phi) = \infty$ for every $y \in F$.

** By $\overline{\text{co}}\{y_n\}$ we denote the closed convex hull of the sequence $\{y_n\}$.*

continuous from above, it is necessary and sufficient that X satisfy condition (EPS) $\wedge$ (HR).

A set $M \subset X$ is called Chebyshev ([1]) if, for every point $x \in X$, the set $T_M x$ is a singleton.

Corollary 1. In a reflexive B-space X satisfying condition (EPS) $\wedge$ (HR), the metric projection onto every convex Chebyshev set is continuous.

Theorem 2. In a reflexive B-space X the following assertions are equivalent:

(a) the metric projection onto every convex closed set is upper semicontinuous;

(b) X satisfies condition (EPS) $\wedge$ (CR);

(c) X satisfies the following condition:

(N) From $\{x_n\} \subset S,\ x_0 \in S,\ \{f_n\} \subset S^*,\ f_0 \in S^*,\ f_n(x_n) = f_0(x_0) = 1,\ x_n \to x_0,\ f_n \to f_0$ and $\rho(x_0, H_0 \cap H_n) \to 0,\ \rho(x_n, H_0 \cap H_n) \to 0$, where $H_0 = \{x : f_0(x) = 1\},\ H_n = \{x :\ f_n(x) = 1\}$, it follows that $x_n \to x_0$.

Corollary 2. In order that, in a strictly convex and reflexive B-space X, the metric projection onto every convex closed set be continuous, it is necessary and sufficient that X satisfy the following condition:

(N_R) From $\{x_n\} \subset S,\ x_0 \in S,\ \{f_n\} \subset S^*,\ f_0 \in S^*,\ f_n(x_n) = f_0(x_0) = 1,\ f_n \to f_0,\ \rho(x_0, H_0 \cap H_n) \to 0,\ \rho(x_n, H_0 \cap H_n) \to 0$, where $H_0(x :\ f_0(x) = 1),\ H_n = \{x :\ f_n(x) = 1\}$, it follows that $x_n \to x_0$.

A set $M \subset X$ is called boundedly weakly bicompact if its intersection with every closed ball is weakly bicompact.

Theorem 3. Let X be a separable B-space. Then, in order that in X the metric projection onto every boundedly weakly bicompact convex set be H-upper semicontinuous, it is necessary and sufficient that X satisfy condition (EPS) $\wedge$ (WHR).

Corollary 3. In a separable H-space X satisfying condition (EPS) $\wedge$ (WHR), the metric projection onto every boundedly weakly bicompact convex Chebyshev set is continuous.

Theorem 4. In a separable B-space X satisfying condition (WCR), the following assertions are equivalent:

(a) the metric projection onto every boundedly weakly bicompact convex set is upper semicontinuous;

(b) X satisfies condition (EPS) $\wedge$ (CR);

(c) X satisfies condition (N).

According to I. Singer ([4]), a B-space X has the Efimov–Stechkin property if it is reflexive and from $\{x_n\} \subset S,\ x_0 \in S,\ x_n \rightharpoonup x_0$ it always follows that $x_n \to x_0$. I. Singer ([4]) showed that in a B-space with the Efimov–Stechkin property the metric projection onto every convex closed set is upper semicontinuous (and hence H-upper semicontinuous); in particular, the metric projection onto every convex Chebyshev set is continuous.

Example. Consider in the space l_2 of real square-summable numerical sequences $x = \{\xi_i\}$ the following norm, equivalent to the original one:

$$\|x\|_{S'} = \inf_{x \in \lambda S'} |\lambda|,$$

where

$$S' = \left\{ x = \{\xi_i\} \in l_2 :\ \sum_{i=1}^{\infty} \xi_i^2 \leq 1,\ |\xi_i| \leq \tfrac{1}{2} \right\}.$$

Denote by $\tilde{l}_2$ the space l_2 with norm $\|x\|_{S'}$. Then $\tilde{l}_2$ is a separable reflexive B-space not possessing the Efimov–Stechkin property (see, for example, ([4])). Moreover, in the space $\tilde{l}_2$ the metric projection onto the closed hyperplane $H = \{x = \{\xi_i\} :\ \xi_1 = \frac{1}{2}\}$ is not upper semicontinu-

from above. However, it can be shown that $\tilde{l}_2$ satisfies condition (EPS) $\wedge$ (HR) and, consequently, by Theorem 1, in it the metric projection onto every convex closed set is H-upper semicontinuous; in particular, the metric projection onto every convex Chebyshev set is continuous.

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Received
31 V 1968

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Note: Figure translations are in progress. See original paper for figures.