ON INDUCED TRANSITIONS OF A RELATIVISTIC ELECTRON IN A HOMOGENEOUS CONSTANT MAGNETIC FIELD

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Abstract

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PHYSICS

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ON INDUCED TRANSITIONS OF A RELATIVISTIC ELECTRON IN A HOMOGENEOUS CONSTANT MAGNETIC FIELD

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1. In considering the interaction of a relativistic electron with a plane monochromatic electromagnetic wave, we choose the wave functions of the electron with a fixed spin projection onto the direction of the field-strength vector of the constant field*. In this case the formulas for transition probabilities assume a comparatively easily interpretable form, and simple selection rules with respect to the quantum number n are obtained.

According to ([2]), the exact expressions for the normalized wave functions and the energy spectrum of a relativistic electron in a constant homogeneous magnetic field have the form**

$$\Psi_{+1} = \frac{\exp i(p_y y + p_z z)}{\Omega^{1/2}[2(\varepsilon^2 + m\varepsilon)]^{1/2}} \begin{pmatrix} (\varepsilon + m)v^{(n)} \\ 0 \\ p_z v^{(n)} \\ -i(eH2n)^{1/2} v^{(n-1)} \end{pmatrix}, \quad (1)$$

$$\Psi_{-1} = \frac{\exp i(p_y y + p_z z)}{\Omega^{1/2}[2(\varepsilon^2 + m\varepsilon)]^{1/2}} \begin{pmatrix} 0 \\ (\varepsilon + m)v^{(n)} \\ i[2eH(n+1)]^{1/2} v^{(n+1)} \\ -p_z v^{(n)} \end{pmatrix};$$

$$\varepsilon^2 = m^2 + p_z^2 + eH(2n - \mu + 1), \quad (2)$$

where $\mathbf{p} = \{p_x, p_y, p_z\}$ is the electron momentum; ε is the electron energy; m is the electron mass; $\mathbf{H}$ is the strength of the constant magnetic field; $n = 0, 1, 2, ...$; $\mu = \pm 1$; $\Omega = L^2$, L is the normalization length; $v^{(n)} = [(eH)^{1/4}/\pi^{1/4} 2^{n/2} (n!)^{1/2}] e^{\xi^2/2} H_n(\xi)$; $\xi = \sqrt{eH}(x - p_y/eH)$; $H_n(\xi)$ is the Hermite polynomial; the magnetic-field strength vector $\mathbf{H}$ is directed along the z-axis.

We describe the plane monochromatic electromagnetic wave classically. In the case of an elliptically polarized wave, $A_4 = 0$, $\mathbf{A} = \mathbf{a}_1 \cos kx + \mathbf{a}_2 \sin kx$, where $kx = \mathbf{kx} - \omega t$, k_i is the 4-momentum of the wave, and $\mathbf{A}$ is the vector potential of the field.

Let us first consider the case when the direction of propagation of the electromagnetic wave coincides with the direction of the magnetic-field strength vector. We direct the x- and y-axes along the vectors $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively; then

$$\mathbf{a}_1 = \{a_1, 0, 0\}, \qquad \mathbf{a}_2 = \{0, a_2, 0\}, \qquad \mathbf{k} = \{0, 0, k\}.$$

We shall be interested in the probabilities of induced one-quantum transitions of electrons from the state n, p_z, μ to the state n', p'_z, μ'***. Po-

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* The treatment in ([1]), where wave functions with a fixed spin projection onto the direction of the kinetic momentum of the electron are used, appears less adequate for the present problem. Apparently, this is the reason for the complexity of the final expressions in ([1]).

** Everywhere a system of units is used in which $\hbar = c = 1$.

*** A prime denotes the final state.

Since μ can take the values ± 1, and each of them corresponds to its own wave function, it is necessary to compute the probabilities of the following transitions:

$$\begin{gathered}(n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = 1),\\ (n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = -1),\\ (n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = -1),\\ (n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = 1).\end{gathered} \tag{3}$$

The matrix elements of the probabilities of the transitions (3) are determined by the expressions*

$$\begin{gathered}(n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = 1)\\ M = iB(2eH)^{1/2}\left[(a_1 \pm a_2)(\varepsilon + m)n'^{1/2}J(n, n'-1) - (a_1 \mp a_2)(\varepsilon' + m)n^{1/2}J(n-1, n')\right],\\ (n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = -1)\\ M = iB(2eH)^{1/2}\left[(a_1 \pm a_2)(\varepsilon' + m)(n+1)^{1/2}J(n+1, n') - (a_1 \mp a_2)(\varepsilon + m)(n'+1)^{1/2}J(n, n'+1)\right],\\ (n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = -1)\\ M = B\left[(\varepsilon' + m)p_z - (\varepsilon + m)p'_z\right]J(n, n')(a_1 \pm a_2),\\ (n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = 1)\\ M = B\left[(\varepsilon + m)p'_z - (\varepsilon' + m)p_z\right]J(n, n')(a_1 \mp a_2),\end{gathered} \tag{4}$$

where

$$B = e\,\frac{2\pi^2}{\Omega}\,\frac{\delta(p_y - p'_y)\delta(p_z - p'_z \pm k)}{(\varepsilon'^2 + m\varepsilon')^{1/2}(\varepsilon^2 + m\varepsilon)^{1/2}}, \tag{5}$$

$$J(n, n') = \int_{-\infty}^{\infty} v^{(n)}(\xi) v^{(n')}(\xi)\,dx = \begin{cases} 0 & \text{for } n \neq n', \\ 1 & \text{for } n = n'. \end{cases} \tag{6}$$

It follows from (4)–(6) that:

for transitions without spin flip $(n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = 1)$ and $(n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = -1)$, the selection rules are $n' = n \pm 1$; for transitions with spin flip $(n,\ p_z,\ \mu = 1 \to n',\ p'_z,\ \mu' = -1)$ and $(n,\ p_z,\ \mu = -1 \to n',\ p'_z,\ \mu' = 1)$, the selection rule is $n = n'$.

(7)

In addition, the conservation laws must be satisfied:

$$\left.\begin{array}{c} \text{energy conservation law } \varepsilon' = \varepsilon \pm \omega; \\ \text{conservation laws for the } z \text{ and } y \text{ components of momentum } p'_z = p_z \pm k; \quad p'_y = p_y. \end{array}\right\} \tag{8}$$

From (7) and (8) it is seen that the following six transitions are allowed:

$$\begin{gathered} 1)\ (n,\ p_z,\ \mu = 1 \to n' = n+1,\ p'_z,\ \mu' = 1), \\ 2)\ (n,\ p_z,\ \mu = -1 \to n' = n+1,\ p'_z,\ \mu' = -1), \\ 3)\ (n,\ p_z,\ \mu = 1 \to n' = n,\ p'_z,\ \mu' = -1), \\ 4)\ (n,\ p_z,\ \mu = 1 \to n' = n-1,\ p'_z,\ \mu' = 1), \\ 5)\ (n,\ p_z,\ \mu = -1 \to n' = n-1,\ p'_z,\ \mu' = -1), \\ 6)\ (n,\ p_z,\ \mu = -1 \to n' = n,\ p'_z,\ \mu' = 1) \end{gathered} \tag{9}$$

under the condition of compatibility of the equations

$$\begin{gathered} \varepsilon' = \varepsilon \pm \omega, \\ \varepsilon^2 = m^2 + p_z^2 + eH(2n - \mu + 1), \\ \varepsilon'^2 = m^2 + (p_z \pm k)^2 + eH(2n' - \mu' + 1). \end{gathered} \tag{10}$$

From conditions (10) it follows that transitions 1, 2, 3 occur with absorption of energy at the frequency $\omega = eH/(\varepsilon - p_z)$ for parallel vectors **k** and **H**, and at the frequency $\omega = eH/(\varepsilon + p_z)$ for antiparallel **k** and **H**, while transitions 4, 5, 6 occur with emission of energy, also at the frequency $\omega = eH/(\varepsilon - p_z)$ for parallel **k** and **H**, and at the frequency $\omega = eH/(\varepsilon + p_z)$ for antiparallel **k** and **H**.

Finally, after integration over the momenta of the final state of the electron, the transition probabilities (9) are determined by the expressions:

$$
\begin{aligned}
w_1 &= \frac{e^2\pi}{4}(n+1)eH\,\frac{\varepsilon+m}{\varepsilon\varepsilon'(\varepsilon'+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon-\omega),\\
w_2 &= \frac{e^2\pi}{4}(n+1)eH\,\frac{\varepsilon'+m}{\varepsilon\varepsilon'(\varepsilon+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon-\omega),\\
w_3 &= \frac{e^2\pi}{8}\frac{\left[(\varepsilon'+m)p_z-(\varepsilon+m)(p_z+k)\right]^2}{\varepsilon\varepsilon'(\varepsilon'+m)(\varepsilon+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon-\omega),\\
w_4 &= \frac{e^2\pi}{4}neH\,\frac{\varepsilon'+m}{\varepsilon\varepsilon'(\varepsilon+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon+\omega),\\
w_5 &= \frac{e^2\pi}{4}neH\,\frac{\varepsilon+m}{\varepsilon\varepsilon'(\varepsilon'+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon+\omega),\\
w_6 &= \frac{e^2\pi}{8}\frac{\left[(\varepsilon'+m)p_z-(\varepsilon+m)(p_z-k)\right]^2}{\varepsilon\varepsilon'(\varepsilon'+m)(\varepsilon+m)}(a_1+a_2)^2\delta(\varepsilon'-\varepsilon+\omega).
\end{aligned}
\tag{11}
$$

2. Let us briefly consider the general case, when the electromagnetic radiation propagates at an arbitrary angle to the direction of the intensity vector of the constant homogeneous magnetic field. In this case we direct the z-axis along $\mathbf{H}$, the x-axis in the plane formed by the vectors $\mathbf{k}$ and $\mathbf{H}$, and the y-axis perpendicular to this plane.

In contrast to the case considered earlier, where the z-components of the matrix elements vanished, here they will, generally speaking, be different from zero, and instead of the quantity $J(n,n')$, determined by (6), we obtain

$$
I(n,n') = \int_{-\infty}^{\infty} v^{(n)}(\xi)v^{(n')}(\xi)e^{\pm ik_x x}\,dx. \tag{12}
$$

The transition probabilities are determined by the expressions:

$$
(n, p_z, \mu = 1 \to n', p_z', \mu' = 1)
$$

$$
\begin{aligned}
w = D\{&2eH(\varepsilon+m)^2 n'|I(n,n'-1)|^2|a_x i - a_y|^2 +\\
&+2eH(\varepsilon'+m)^2 n|I(n',n-1)|^2|a_x i + a_y|^2 - 4eH(\varepsilon+m)(\varepsilon'+m)\times\\
&\times n'^{1/2}n^{1/2}\,\mathrm{Re}\big[I(n,n'-1)I^*(n',n-1)(a_x i - a_y)(a_y^* - i a_x^*)\big] +\\
&+|a_z|^2\left[(\varepsilon'+m)p_z+(\varepsilon+m)p_z'\right]^2|I(n,n')|^2 + (2eH)^{1/2}\left[(\varepsilon'+m)p_z +\right.\\
&\left.+(\varepsilon+m)p_z'\right]\big[(\varepsilon+m)n'^{1/2}2\,\mathrm{Re}\big[I^*(n,n')I(n,n'-1)a_z^*(a_x i - a_y)\big]\\
&-(\varepsilon'+m)n^{1/2}2\,\mathrm{Re}\big[I(n,n')I^*(n',n-1)a_z(a_y^*-ia_x^*)\big]\big]\big\},
\end{aligned}
$$

$$
(n, p_z, \mu = -1 \to n', p_z', \mu' = -1) \tag{13}
$$

$$w = D\{2eH(\varepsilon' + m)^2(n+1)|I(n', n+1)|^2|a_x i - a_y|^2 +$$
$$+ 2eH(\varepsilon + m)^2(n'+1)|I(n, n'+1)|^2|a_x i + a_y|^2 -$$
$$- 4eH(\varepsilon + m)(\varepsilon' + m)(n'+1)^{1/2}(n+1)^{1/2}\operatorname{Re}[I^*(n, n'+1)I(n', n+1)\times$$
$$\times(a_x i - a_y)(a_y^* - i a_x^*)] + |a_z|^2|I(n, n')|^2\left[(\varepsilon' + m)p_z + (\varepsilon + m)p_z'\right]^2 +$$
$$+ (2eH)^{1/2}\left[(\varepsilon' + m)p_z + (\varepsilon + m)p_z'\right]\left[(\varepsilon' + m)(n+1)^{1/2} 2\operatorname{Re}[I^*(n, n')\times\right.$$
$$\times I(n', n+1)(a_x i - a_y)a_z^*] - (\varepsilon + m)(n'+1)^{1/2}\times$$
$$\left.\times 2\operatorname{Re}\left[I(n, n')I^*(n, n'+1)a_z(a_y^* - i a_x^*)\right]\right\},$$

$$(n, p_z, \mu = -1 \to n', p_z', \mu' = 1)$$

$$w = D\{|(\varepsilon + m)p_z' - (\varepsilon' + m)p_z|^2|I(n, n')|^2|a_x - i a_y|^2 +$$
$$+ 2eH|a_z|^2[(\varepsilon' + m)^2(n+1)|I(n', n+1)|^2 + (\varepsilon + m)^2 n'|I(n, n'-1)|^2 -$$
$$- 2(\varepsilon + m)(\varepsilon' + m)(n+1)^{1/2}n'^{1/2}\operatorname{Re}[I(n, n'-1)I^*(n', n+1)]] +$$
$$+ (2eH)^{1/2}[(\varepsilon + m)p_z' - (\varepsilon' + m)p_z][2(\varepsilon' + m)(n+1)^{1/2}\operatorname{Re}[i a_z(a_x^* + i a_y^*)\times$$
$$\times I^*(n, n')I(n', n+1)] - 2(\varepsilon + m)n'^{1/2}\operatorname{Re}[i a_z(a_x^* + i a_y^*)\times$$
$$\times I(n, n'-1)I^*(n, n')]]\},$$

$$(n, p_z, \mu = 1 \to n', p_z', \mu' = -1)$$

$$w = D\{|(\varepsilon + m)p_z' - (\varepsilon' + m)p_z|^2|I(n, n')|^2|a_x + i a_y|^2 +$$
$$+ 2eH|a_z|^2[(\varepsilon' + m)^2 n|I(n', n-1)|^2 + (\varepsilon + m)^2(n'+1)|I(n, n'+1)|^2 -$$
$$- 2(\varepsilon' + m)(\varepsilon + m)(n'+1)^{1/2}n^{1/2}\operatorname{Re}[I(n', n-1)I^*(n, n'+1)]] +$$
$$+ (2eH)^{1/2}[(\varepsilon' + m)p_z - (\varepsilon + m)p_z']\, 2(\varepsilon' + m)n^{1/2}\times$$
$$\times \operatorname{Re}[I(n', n-1)I^*(n, n')i a_z(a_x^* - i a_y^*)] -$$
$$- (\varepsilon + m)(n'+1)^{1/2} 2\operatorname{Re}[I(n, n'+1)I^*(n, n')i a_z(a_x^* - i a_y^*)]\},$$

where

$$D = \frac{e^2\pi}{\varepsilon\varepsilon'(\varepsilon' + m)(\varepsilon + m)}\delta(\varepsilon' - \varepsilon - \omega); \qquad \mathbf{a} = \{a_x, a_y, a_z\} = \frac{1}{2}(\mathbf{a}_1 - i\mathbf{a}_2).$$

These formulas have been written for the case of absorption. In the case of emission, in them, instead of the projections on the coordinate axes of the vector **a**, one must substitute the projections of the vector $\mathbf{a}^*$, and instead of the projections of the vector $\mathbf{a}^*$, the projections of the vector **a**. In the argument of the δ-function one must replace $-\omega$ by $+\omega$.

According to (4) we have

$$I(n,n') = \int_{-\infty}^{\infty} v^{(n)}(\xi) v^{(n')}(\xi) e^{\pm i k_x x} =$$

$$= \frac{\exp\{-\alpha^2/4 \pm i k_x p_y / eH\}}{2^{(n+n')/2}(n!n'!)^{1/2}} \begin{cases} 2^n n'! \left(\pm \frac{i\alpha}{2}\right)^{n-n'} L_n^{n-n'}\left(\dfrac{\alpha^2}{2}\right), & n' \leqslant n, \\ 2^{n'} n! \left(\pm \frac{i\alpha}{2}\right)^{n'-n} L_{n'}^{n'-n}\left(\dfrac{\alpha^2}{2}\right), & n' \geqslant n, \end{cases} \tag{14}$$

where $\alpha = k_x/\sqrt{eH}$; $L_n^{n-n'}$ and $L_{n'}^{n'-n}$ are generalized Laguerre polynomials.

The generalized Laguerre polynomials may be represented in the form (see (3))

$$L_n^{n-n'}\left(\frac{\alpha^2}{2}\right) = \sum_{\nu=0}^{n} \binom{n+n-n'}{n-\nu} \frac{(-\alpha^2/2)^\nu}{\nu!}. \tag{15}$$

Using the expansion (15), we obtain that for $\alpha = 0$, $I(n,n') = J(n,n')$, and the already known selection rules are $n' = n \pm 1, 0$.

For $\alpha \ll 1$, expanding (14) in a series in powers of α, we see that as the degree of α increases, the number of levels to which a transition from the initial energy level is allowed increases.

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Note: Figure translations are in progress. See original paper for figures.