TENSOR INVARIANTS OF SUBGROUPS OF THE LORENTZ GROUP

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Abstract

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TENSOR INVARIANTS OF SUBGROUPS OF THE LORENTZ GROUP

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For every subgroup $\mathcal{L}_k$ of the Lorentz group $\mathcal{L}$ that can be specified by the requirement of invariance of some finite set of tensors $H_{(1)}, \dots, H_{(n)}$, any other tensor invariant T of it, of arbitrary rank, can be represented in the form of a linear combination of tensors $T_{(1)}, \dots, T_{(p)}$ constructed by means of the operations of tensor multiplication and contraction from the basic tensors $H_{(1)}, \dots, H_{(n)}$:

$$T = k_1 T_{(1)} + \dots + k_p T_{(p)},$$

where the tensors $T_{(1)}, \dots, T_{(p)}$ form a basis of the linear space of tensors of rank r invariant with respect to the group $\mathcal{L}_k$, the dimension of which is denoted by p.*

Thus, the construction of tensor invariants for subgroups of the Lorentz group $\mathcal{L}$ reduces to finding the basic tensors $\{H_m\}$ for all subgroups $\mathcal{L}_k$ of the Lorentz group $\mathcal{L}$ that can be specified with the aid of tensors. All these subgroups $\mathcal{L}_k$ are Lie subgroups of the Lorentz group with a finite component group, whose classification is given in ([6]). The method for choosing all subgroups $\mathcal{L}_k$ and obtaining their basic tensors $\{H_m\}$ is indicated in ([5]).

Table 1 lists all subgroups $\mathcal{L}_k$ of the Lorentz group $\mathcal{L}$ that are specified with the aid of tensors, and for all these subgroups the corresponding sets of tensors are written out. In some cases the group can be specified by a smaller number of invariant tensors; however, the tensors indicated are sufficient to obtain the entire algebra of tensor invariants.

At the beginning of each series of formulas the infinitesimal operators of the maximal connected subgroup $\mathcal{G}_l$ of the group $\mathcal{L}_k$ under consideration are indicated, while the symbols of A. V. Shubnikov** are used to denote its finite component group ([6]).

The infinitesimal operators X_q of the proper Lorentz group of transformations of the variables x^i $(i = 1, 2, 3, 4)$, which leave invariant the quadratic form $x^{12} + x^{22} + x^{32} - x^{42}$, have the form

$$X_1 = x^2\partial_3 - x^3\partial_2, \quad X_2 = x^3\partial_1 - x^1\partial_3, \quad X_3 = x^1\partial_2 - x^2\partial_1,$$

$$X_4 = x^1\partial_4 + x^4\partial_1, \quad X_5 = x^2\partial_4 + x^4\partial_2, \quad X_6 = x^3\partial_4 + x^4\partial_3,$$

where $\partial_i = \partial/\partial x^i$.

The vectors e_1, e_2, e_3, e_4 form an orthonormal basis. Products and powers of the vectors e_i are understood as tensor products, $e_{ij\dots k} = e_i e_j \dots e_k$. The symbols [], ( ) attached to indices denote alternation and symmetrization.

* Basic tensors $\{H_{(m)}\}$ and tensor bases $\{T_{(s)}\}$ of the first, second, third, and fourth ranks for subgroups of the group $O(3)$ were established in ([1]-[4]).

** n denotes a rotation about the x^3 axis through the angle $2\pi/n$; $1:2$ denotes a rotation about the x^2 axis through the angle π; reflection of x^3 is denoted by the symbol $:m$, or by a bar over the reflected x^3; reflection of x^2 by the symbol $\cdot m$, reflection of x^4 by a subscript bar, $\cdot m'$ denotes reflection of x^1, $1:2'$ denotes rotation about the x^1 axis through the angle π. 3/2, 3/4, and 3/5 denote the symmetry groups of the tetrahedron, octahedron, and icosahedron.

Table 1

Subgroup	Parameter	Tensor invariants	Subgroup	Parameter	Tensor invariants
$\mathcal{G}_1(X_1, X_2, X_3, X_4, X_5, X_6)$		$\mathfrak{g},\ E$		$1\cdot\underline{m}$	$\mathfrak{g},\ E,\ \mathbf{e}_4^2,\ \mathbf{e}_3$
	$\frac{2}{2}\cdot m$	$\mathfrak{g}$		$1\cdot m$	$\mathfrak{g},\ \mathbf{e}_4^2,\ \mathbf{e}_3$
$\mathcal{G}_2(X_1, X_2, X_3)$				$\frac{2}{2}\cdot m$	$\mathfrak{g},\ \mathbf{e}_3^2,\ \mathbf{e}_4$
	1	$\mathfrak{g},\ E,\ \mathbf{e}_4$		$\frac{2}{\underline{2}}\cdot m$	$\mathfrak{g},\ \mathbf{e}_{[34]},\ \mathbf{e}_4^2$
	$\frac{\underline{2}}{2}$	$\mathfrak{g},\ \mathbf{e}_4$		$\frac{2}{2}\cdot m$	$\mathfrak{g},\ E,\ \mathbf{e}_3^2,\ \mathbf{e}_4^2$
	$\frac{1}{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]}$		$\frac{2}{\underline{2}}\cdot m$	$\mathfrak{g},\ \mathbf{e}_{[124]},\ \mathbf{e}_4^2$
	$\frac{\underline{2}}{2}$	$\mathfrak{g},\ E,\ \mathbf{e}_4^2$		$\frac{2}{\underline{2}}\cdot m\times\underline{1}$	$\mathfrak{g},\ \mathbf{e}_3^2,\ \mathbf{e}_4^2$
	$\frac{\underline{2}}{2}\times\underline{1}$	$\mathfrak{g},\ \mathbf{e}_4^2$	$\mathcal{G}_8(X_6)$		
$\mathcal{G}_3(X_3, X_4, X_5)$				$n:\underline{m}$	$\mathfrak{g},\ E,\ D_{nh},\ \mathbf{e}_{[12]}$
	$1\cdot\underline{m}$	$\mathfrak{g},\ E,\ \mathbf{e}_3$		$\underline{m}\cdot n:\underline{m}$	$\mathfrak{g},\ E,\ D_{nh},\ \mathbf{e}_2^2\ (n = 1)$
	$1\cdot m$	$\mathfrak{g},\ \mathbf{e}_3$		$m\cdot n:\underline{m}$	$\mathfrak{g},\ D_{nh},\ \mathbf{e}_{[34]}$
	$\frac{\underline{2}}{2}\cdot m$	$\mathfrak{g},\ E,\ \mathbf{e}_3^2$		$n:\underline{m}\times\underline{1}$	$\mathfrak{g},\ D_{nh},\ \mathbf{e}_{[12]}$
	$\frac{2}{\underline{2}}\cdot m$	$\mathfrak{g},\ \mathbf{e}_{[124]}$		$m\cdot n:\underline{m}\times\underline{1}$	$\mathfrak{g},\ D_{nh},\ \mathbf{e}_2^2\ (n = 1)$

Subgroup	Parameter	Tensor invariants	Subgroup	Parameter	Tensor invariants
	$\frac{2}{\underline{2}}\cdot m\times\underline{1}$	$\mathfrak{g},\ \mathbf{e}_3^2$		$2n:\underline{m}$	$\mathfrak{g},\ \mathbf{e}_{[34]}D_{nh},\ \mathbf{e}_{[12]}$
$\mathcal{G}_4(X_3,\ X_4-X_2,\ X_1+X_5)$				$m\cdot 2n:\underline{m}$	$\mathfrak{g},\ \mathbf{e}_{[34]}D_{nh}$
	1	$\mathfrak{g},\ E,\ \mathbf{e}_3+\mathbf{e}_4$	$\mathcal{G}_9(X_4-X_2)$		
	$1:\underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]}+\mathbf{e}_{[124]}$		1	$\mathfrak{g},\ E,\ \mathbf{e}_2,\ \mathbf{e}_3+\mathbf{e}_4$
	$\underline{2}$	$\mathfrak{g},\ E,\ (\mathbf{e}_3+\mathbf{e}_4)^2$		2	$\mathfrak{g},\ E,\ \mathbf{e}_2^2,\ \mathbf{e}_3+\mathbf{e}_4$
	$1\cdot m$	$\mathfrak{g},\ \mathbf{e}_3+\mathbf{e}_4$		$1:\underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]}+\mathbf{e}_{[124]},\ \mathbf{e}_2$
	$\frac{1}{2}\cdot m$	$\mathfrak{g},\ (\mathbf{e}_3+\mathbf{e}_4)^2$		$\underline{2}$	$\mathfrak{g},\ E,\ \mathbf{e}_2^2,\ \mathbf{e}_2(\mathbf{e}_3+\mathbf{e}_4)$
$\mathcal{G}_5(X_3, X_6)$				$\frac{2}{\underline{2}}:\underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]}+\mathbf{e}_{[124]},\ \mathbf{e}_2^2$
	$\underline{2}$	$\mathfrak{g},\ E,\ \mathbf{e}_{[12]}$		$1:\underline{m}$	$\mathfrak{g},\ E,\ (\mathbf{e}_3+\mathbf{e}_4)^2\mathbf{e}_2$
	$\frac{2}{2}\cdot m$	$\mathfrak{g},\ E,\ \mathbf{e}_3^2-\mathbf{e}_4^2$		$2:\underline{m}$	$\mathfrak{g},\ E,\ \mathbf{e}_2^2,\ (\mathbf{e}_3+\mathbf{e}_4)^2$
	$\frac{2}{2}\times\underline{1}$	$\mathfrak{g},\ \mathbf{e}_{[12]}$		$1\cdot m$	$\mathfrak{g},\ \mathbf{e}_{[134]},\ \mathbf{e}_3+\mathbf{e}_4$
	$\underline{2}\cdot m$	$\mathfrak{g},\ \mathbf{e}_{[34]}$		$2\cdot m$	$\mathfrak{g},\ \mathbf{e}_2^2,\ \mathbf{e}_3+\mathbf{e}_4$
	$\underline{2}\cdot m\times\underline{1}$	$\mathfrak{g},\ \mathbf{e}_3^2-\mathbf{e}_4^2$		$m'\cdot 1:\underline{m}$	$\mathfrak{g},\ (\mathbf{e}_3+\mathbf{e}_4)^2,\ \mathbf{e}_2$
$\mathcal{G}_6(X_4-X_2,\ X_1+X_5)$				$m\cdot 1:\underline{m}$	$\mathfrak{g},\ \mathbf{e}_{[134]},\ (\mathbf{e}_3+\mathbf{e}_4)^2$
	n	$\mathfrak{g},\ E,\ \mathbf{e}_3+\mathbf{e}_4,\ \omega_n$		$m\cdot 2:\underline{m}$	$\mathfrak{g},\ \mathbf{e}_2^2,\ (\mathbf{e}_3+\mathbf{e}_4)^2$
	$\underline{2n}$	$\mathfrak{g},\ E,\ (\mathbf{e}_3+\mathbf{e}_4)^2,\ \omega_n$ (n odd), $\omega_{2n},\ (\mathbf{e}_3+\mathbf{e}_4)\omega_n$ (n even)		$\underline{2}\cdot m$	$\mathfrak{g},\ \mathbf{e}_2^2,\ \mathbf{e}_{[14]}+\mathbf{e}_{[13]}$
	$n:\underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]}+\mathbf{e}_{[124]},\ \omega_n$		$1:\underline{2}'$	$\mathfrak{g},\ \mathbf{e}_{[134]},\ \mathbf{e}_2(\mathbf{e}_3+\mathbf{e}_4)$
	$n:\underline{m}$	$\mathfrak{g},\ E,\ (\mathbf{e}_3+\mathbf{e}_4)^2,\ \omega_{2n},\ (\mathbf{e}_3+\mathbf{e}_4)\omega_n$ (n odd), ω_n (n even)		$1\cdot m'$	$\mathfrak{g},\ \mathbf{e}_2,\ \mathbf{e}_3+\mathbf{e}_4$

Subgroup	Parameter	Tensor invariants	Subgroup	Parameter	Tensor invariants
	$n \cdot m$	$\mathfrak{g},\ \mathbf{e}_3 + \mathbf{e}_4,\ \omega_n$		$\underline{2} \cdot m'$	$\mathfrak{g},\ \mathbf{e}_2^2,\ \mathbf{e}_2(\mathbf{e}_3 + \mathbf{e}_4)$
	$m \cdot n : \underline{m}$	$\mathfrak{g},\ (\mathbf{e}_3 + \mathbf{e}_4)^2,\ \omega_{2n},\ (\mathbf{e}_3 + \mathbf{e}_4)\omega_n$ (n odd), ω_n (n even)	**Finite groups**		
	$\underline{2n} \cdot m$	$\mathfrak{g},\ (\mathbf{e}_3 + \mathbf{e}_4)^2,\ \omega_n$ (n odd), $\omega_{2n},\ (\mathbf{e}_3 + \mathbf{e}_4)\omega_n$ (n even)		n	$\mathfrak{g},\ E,\ D_{nh},\ \mathbf{e}_3,\ \mathbf{e}_4$
$\mathcal{G}_7(X_3)$				$\underline{2n}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_4 D_{nh},\ \mathbf{e}_3$
	1	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_3\mathbf{e}_4$		$n \times \underline{1}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ D_{nh},\ \mathbf{e}_3$
	$\underline{1}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_3$		$n : \underline{2}$	$\mathfrak{g},\ E,\ \mathbf{e}_4,\ \mathbf{e}_3 D_{nh}$ (n odd), D_{nh} (n even)
	$1 : \underline{2}$	$\mathfrak{g},\ E,\ \mathbf{e}_3^2,\ \mathbf{e}_4$		$n : \underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[123]},\ \mathbf{e}_3\mathbf{e}_4,\ \mathbf{e}_3 D_{nh}$ (n even), D_{nh} (n ev
	$1 : 2$	$\mathfrak{g},\ \mathbf{e}_{[123]},\ \mathbf{e}_{[34]}$		$\underline{2n} : 2$	$\mathfrak{g},\ \mathbf{e}_{[123]},\ \mathbf{e}_3\mathbf{e}_4 D_{nh}$ (n odd), $\mathbf{e}_4 D_{nh}$ (n ever
	$1 : \underline{2}$	$\mathfrak{g},\ \mathbf{e}_3\mathbf{e}_{[12]}$		$n : \underline{2} \times \underline{1}$	$\mathfrak{g},\ \mathbf{e}_{[123]},\ \mathbf{e}_3 D_{nh}$ (n odd), D_{nh} (n even)
	$\underline{2}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_4$		$\underline{2n}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_4$
	2	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_{[34]},\ \mathbf{e}_4^2$		$\frac{\underline{2n}}{2n}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_3\mathbf{e}_4$
	$\frac{\underline{2}}{2} \times \underline{1}$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_4^2$		$\underline{2n}\times$	$\mathfrak{g},\ \mathbf{e}_{[12]},\ \mathbf{e}_3 D_{nh}$
	$1 \cdot m$	$\mathfrak{g},\ \mathbf{e}_{[34]},\ \mathbf{e}_4$			

(continued)

$n : m$	$g,\ \mathbf{e}_{[12]},\ D_{nh},\ \mathbf{e}_4$	$3/2$	$g,\ \mathbf{e}_{[123]},\ T_d,\ \mathbf{e}_4$
$n : \underline{m}$	$g,\ \mathbf{e}_{[12]},\ D_{nh},\ \mathbf{e}_3\mathbf{e}_4$	$3/2 \times \underline{1}$	$g,\ \mathbf{e}_{[123]},\ T_d$
$\underline{2n} : m$	$g,\ \mathbf{e}_{[12]},\ \mathbf{e}_4 D_{nh}$	$\overline{6}/2$	$g,\ T_h,\ \mathbf{e}_4$
$\underline{2n} : \underline{m}$	$g,\ \mathbf{e}_{[12]},\ \mathbf{e}_3\mathbf{e}_4 D_{nh}$	$\overline{6}/2$	$g,\ E,\ T_h$
$n : m \times \underline{1}$	$g,\ \mathbf{e}_{[12]},\ D_{nh},\ \mathbf{e}_3^2$	$\overline{6}/2 \times \underline{1}$	$g,\ T_h$
$n \cdot m$	$g,\ P_{nh},\ \mathbf{e}_3,\ \mathbf{e}_4$	$3/4$	$g,\ \mathbf{e}_{[123]},\ O_h,\ \mathbf{e}_4$
$n \cdot \underline{m}$	$g,\ D_{nh},\ \mathbf{e}_4\mathbf{e}_{[12]},\ \mathbf{e}_3$	$3/\overline{4}$	$g,\ \mathbf{e}_{[123]},\ \mathbf{e}_4 T_d$
$\underline{2n} \cdot m$	$g,\ \mathbf{e}_4 D_{nh},\ \mathbf{e}_3$	$3/\overline{4} \times \underline{1}$	$g,\ \mathbf{e}_{[123]},\ O_h$
$n \cdot m \times \underline{1}$	$g,\ D_{nh},\ \mathbf{e}_3,\ \mathbf{e}^2$ ($n = 1$)	$3/\overline{4}$	$g,\ T_d,\ \mathbf{e}_4$
$m \cdot n : m$	$g,\ D_{nh},\ \mathbf{e}_4,\ \mathbf{e}_2^2$ ($n = 1$)	$3/\overline{4}$	$g,\ E,\ T_d$
$\underline{m} \cdot n : m$	$g,\ D_{nh},\ \mathbf{e}_4\mathbf{e}_{[12]}$	$3/\overline{4} \times \underline{1}$	$g,\ T_d$
$\underline{m} \cdot n : \underline{m}$	$g,\ E,\ D_{nh},\ \mathbf{e}_4^2,\ \mathbf{e}_2^2$ ($n \neq 1$)	$\overline{6}/4$	$g,\ O_h,\ \mathbf{e}_4$
$m \cdot n : \underline{m}$	$g,\ D_{nh},\ \mathbf{e}_3\mathbf{e}_4$	$\overline{6}/4$	$g,\ \mathbf{e}_4 T_h$
$m \cdot \underline{2n} : m$	$g,\ \mathbf{e}_4 D_{nh},\ \mathbf{e}_2^2$ ($n = 1$)	$\overline{6}/\overline{4}$	$g,\ E,\ O_h$

$m \cdot \underline{2n} : \underline{m}$	$g,\ \mathbf{e}_3\mathbf{e}_4 D_{nh}$	$\overline{6}/\overline{4}$	$g,\ \mathbf{e}_4 T_d$
$m \cdot n : m \times \underline{1}$	$g,\ D_{nh},\ \mathbf{e}_3^2,\ \mathbf{e}_2^2\ (n = 1)$	$\overline{6}/\overline{4} \times \underline{1}$	$g,\ O_h$
$\overline{2n} \cdot m$	$g,\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_4$	$3/5$	$g,\ \mathbf{e}_{[123]},\ Y_h,\ \mathbf{e}_4$
$\overline{2n} \cdot m$	$g,\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_3\mathbf{e}_4$	$3/5 \times \underline{1}$	$g,\ \mathbf{e}_{[123]},\ Y_h$
$\overline{2n} \cdot \underline{m}$	$g,\ E,\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_2^2\ (n = 1)$	$3/\overline{10}$	$g,\ Y_h,\ \mathbf{e}_4$
$\overline{2n} \cdot \underline{m}$	$g,\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_4\mathbf{e}_{[12]}$	$3/\overline{10}$	$g,\ \mathbf{e}_4 Y_h$
$\overline{2n} \cdot m \times \underline{1}$	$g,\ \mathbf{e}_3 D_{nh},\ \mathbf{e}_2^2\ (n = 1)$	$3/\overline{10} \times \underline{1}$	$g,\ Y_h$

For some tensors special notations have been introduced: $g = e_1^2 + e_2^2 + e_3^2 - e_4^2$ is the metric tensor specifying the Lorentz group $\mathcal{L}$; $E = e_{[1234]}$; $O_h = e_1^4 + e_2^4 + e_3^4$; $T_d = e_{(123)}^4$; $T_h = e_1^2 e_2^2 - e_2^2 e_1^2 + e_2^2 e_3^2 - e_3^2 e_2^2 + e_3^2 e_1^2 - e_1^2 e_3^2$; $Y_h = 5e_1^6 + e_1^4 e_2^2 + e_1^2 e_2^4 + 5e_2^6 - e_1^5 e_3 + e_1^3 e_2^2 e_3 + e_1 e_2^4 e_3 + 2e_1^2 e_3^4 + 2e_2^2 e_3^4 + 2e_3^6$, where the square bracket above denotes summation over all isomers (differing only in the order of the factors); $D_{nh} = \mathrm{Re}(e_1 + ie_2)^n = e_1^n - e_1^{n-2} e_2^2 + e_1^{n-4} e_2^4 - ...$; $\omega_n = \mathrm{Re}(e_{[31]} + e_{[41]} + ie_{[32]} + ie_{[42]})^n$.

Using the known theorems and formulas of the theory of finite-dimensional representations of classical groups (see, for example, ([7])), one can, by means of the methods of analytic continuation, induction, and the theory of characters of compact groups, derive simple formulas for the dimension $p_i = p_i(r, \mathcal{G}_i)$ of the space of invariant tensors of rank r for all connected subgroups $\mathcal{G}_i$ of the Lorentz group: $p_1 = \left(C_r^{r/2}(r/2+1)\right)^2$ for even r, and $p_1 = 0$ for odd r; $p_2 = p_3 = p_4 = C_{2r}^r/(r+1)$, $p_5 = (2C_{r-1}^{r/2})^2$ for even r, and $p_5 = 0$ for odd r; $p_6 = (C_r^{[r/2]})^2$, $p_7 = p_8 = p_9 = C_{2r}^r$.

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Note: Figure translations are in progress. See original paper for figures.